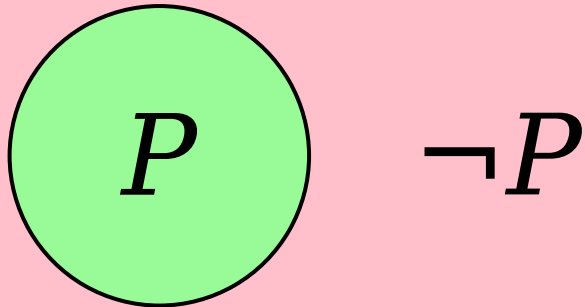


CS103, AUTUMN 2026

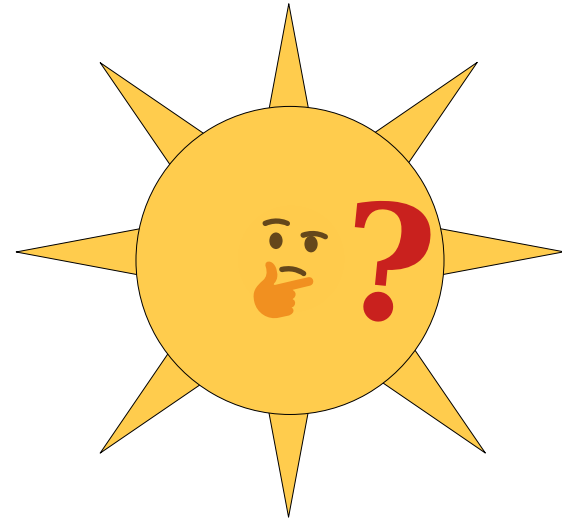


Lecture 02: **Indirect Proofs**

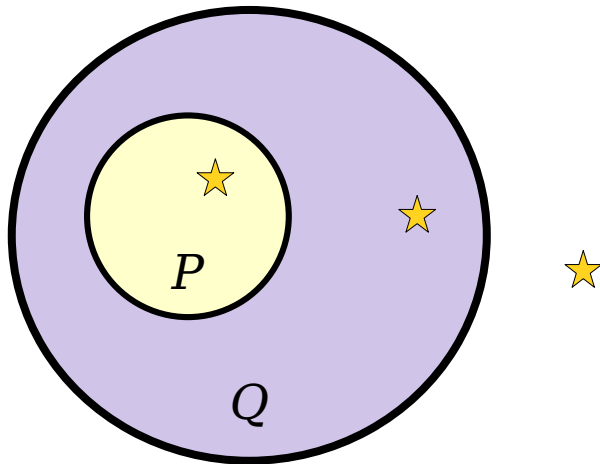
Indirect Proofs: A Story in Four Acts



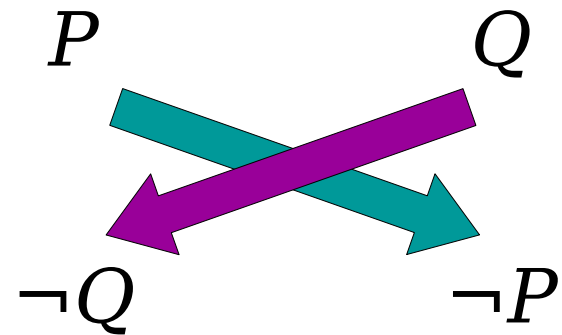
Logical Negation



Proof by Contradiction

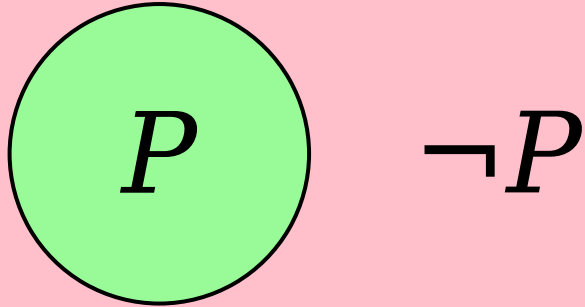


Logical Implication

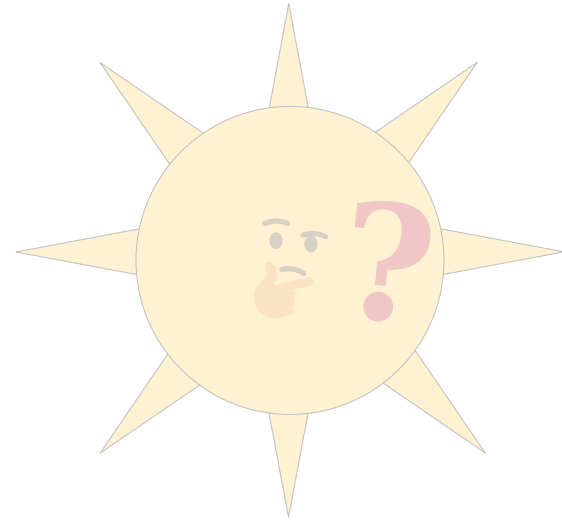


Proof by Contrapositive

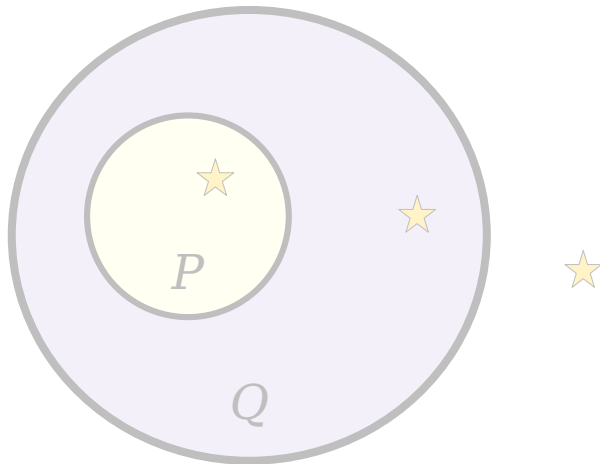
Indirect Proofs: A Story in Four Acts



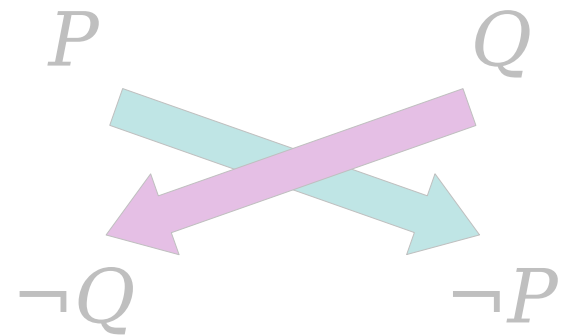
Logical Negation



Proof by Contradiction



Logical Implication



Proof by Contrapositive

Act I

Logical Negation

Propositions and Negations

- A **proposition** is a statement that is either true or false.
- Some examples:
 - If n is an even integer, then n^2 is an even integer.
 - $\emptyset = \mathbb{R}$.
- The **negation** of a proposition X is a proposition that is true when X is false and is false when X is true.
- For example, consider the proposition “it is snowing outside.”
 - Its negation is “it is not snowing outside.”
 - Its negation is *not* “it is sunny outside.” ⚠
 - Its negation is *not* “we’re in the Bay Area.” ⚠

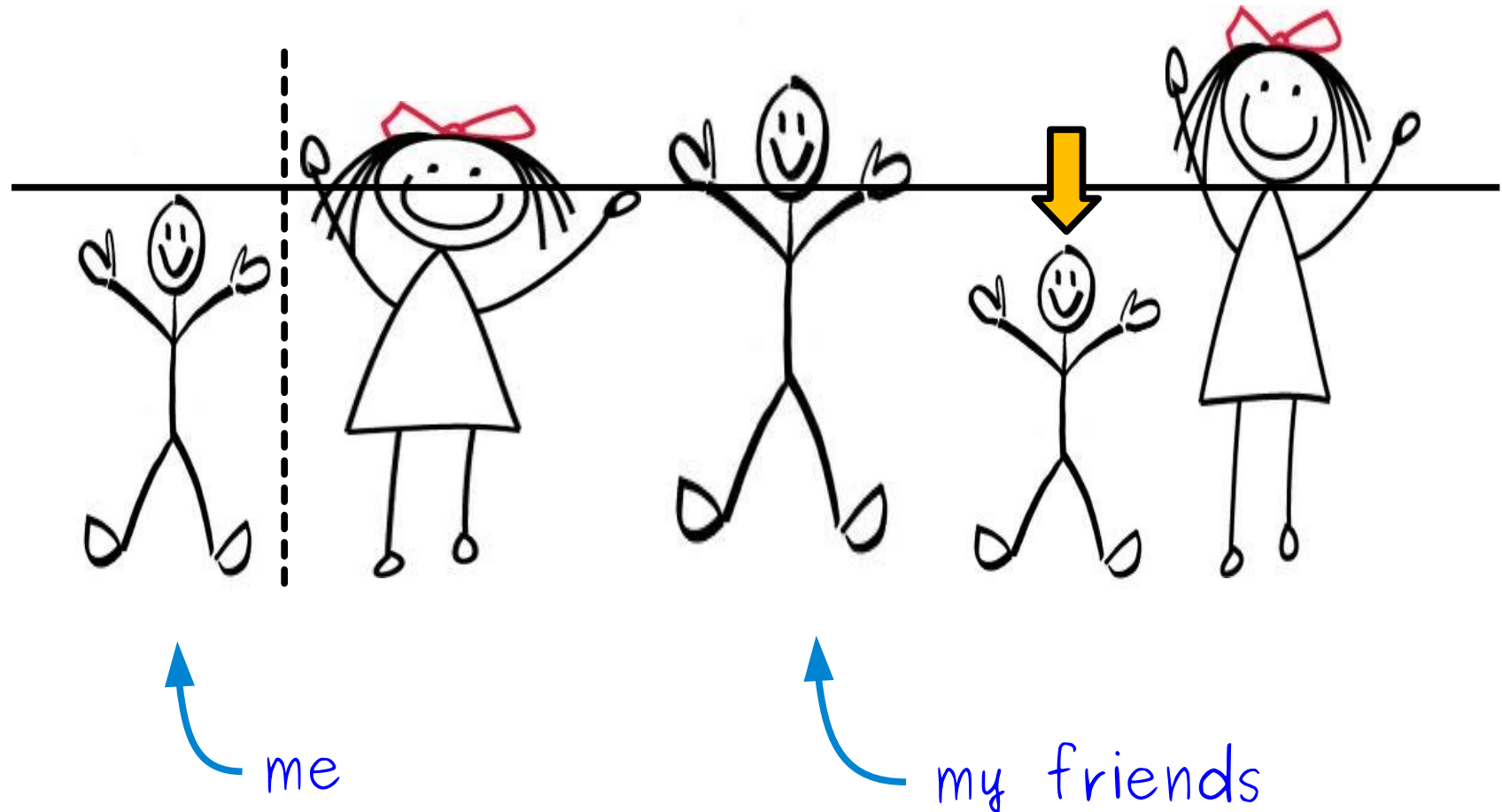
Propositions and Negations

- A **proposition** is a statement that is either true or false.
- Some examples:
 - If n is an integer, $n^2 \geq 0$.
 - $\emptyset = \mathbb{R}$.
- The **negation** of a proposition is a statement that is true when X is false and is false when X is true.
- For example, consider the proposition “it is snowing outside.”
 - Its negation is “it is not snowing outside.”
 - Its negation is *not* “it is sunny outside.” ⚠
 - Its negation is *not* “we’re in the Bay Area.” ⚠

So, how do we find the negation of a proposition?

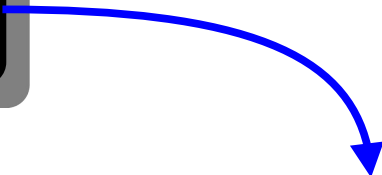
Negating Statements

“All my friends are taller than me.”

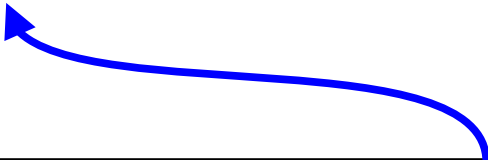


Negating Statements

Every *P* is a *Q*.



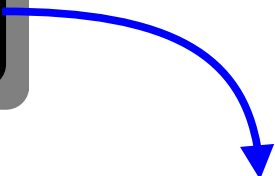
The negation of a ***universal*** statement
is an ***existential*** statement.



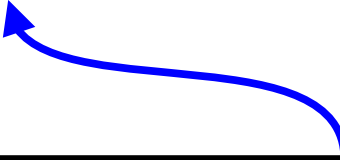
There is a *P* that is not a *Q*.

Negating Statements

For all x , $P(x)$ is true.



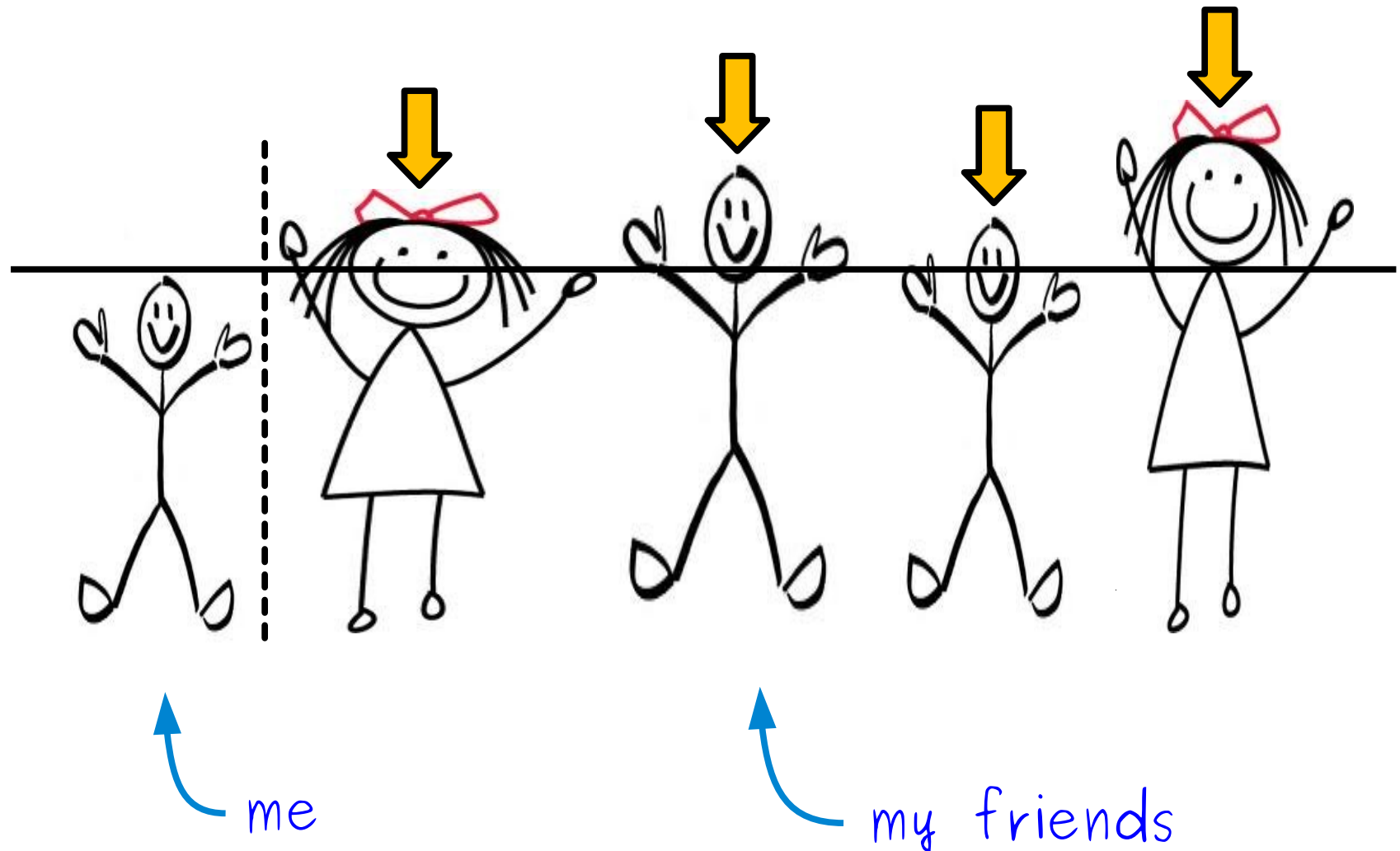
The negation of a ***universal*** statement
is an ***existential*** statement.



There exists an x where $P(x)$ is false.

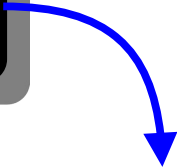
Negating Statements

“Some friend is shorter than me.”

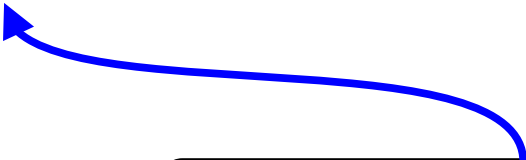


Negating Statements

There exists a *P* that is a *Q*.



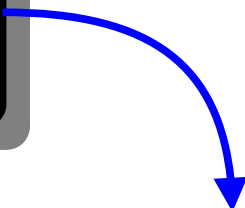
The negation of an *existential* statement
is a *universal* statement.



Every *P* is not a *Q*.

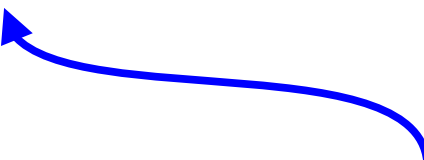
Negating Statements

There exists an x where
 $P(x)$ is true.



The negation of an ***existential*** statement
is a ***universal*** statement.

For all x ,
 $P(x)$ is false.



Negating Statements

- Your turn! What's the negation of the following statement?

**“Every brown dog
loves every orange cat.”**

Negating Statements

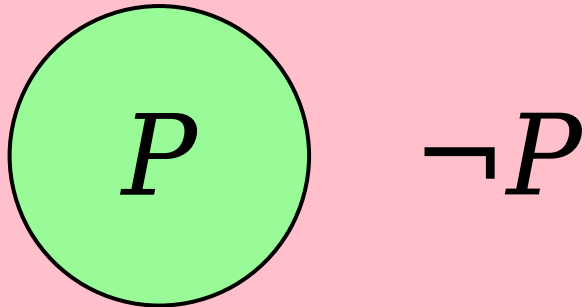
- Your turn! What's the negation of the following statement?

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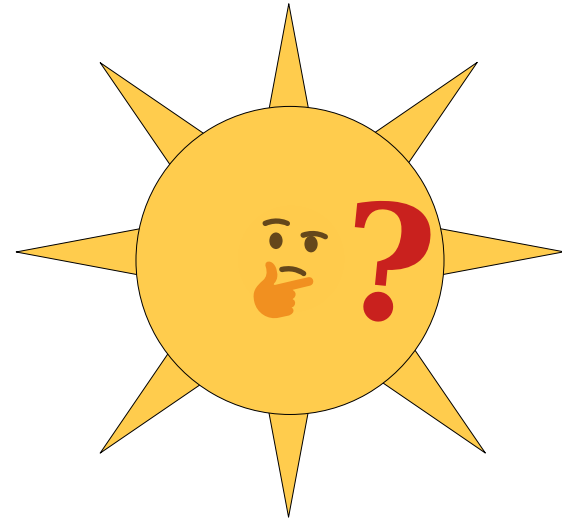
- Answer:

**“There is a brown dog
that doesn't love
some orange cat.”**

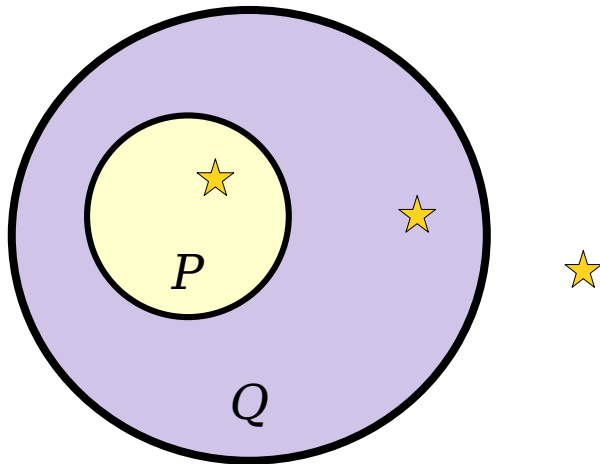
Indirect Proofs: A Story in Four Acts



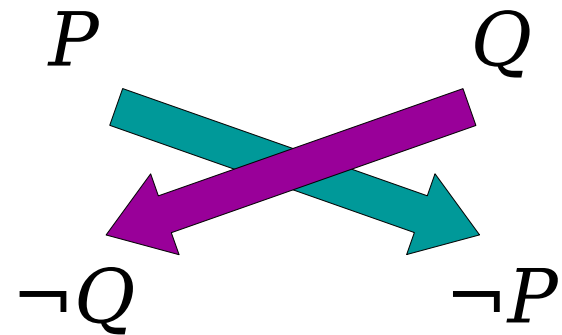
Logical Negation



Proof by Contradiction

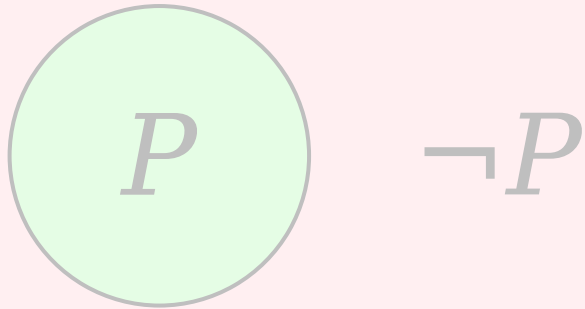


Logical Implication

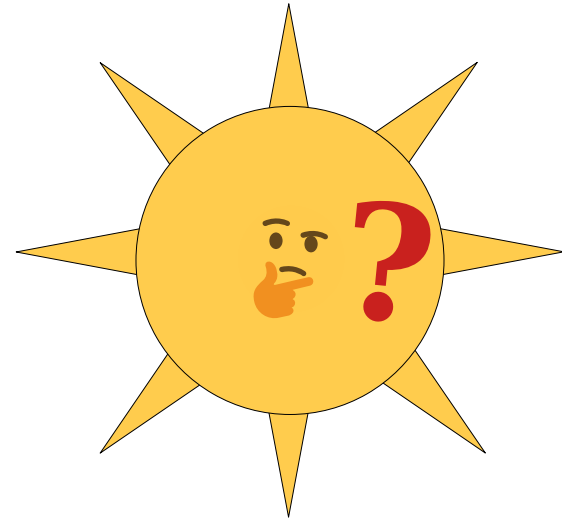


Proof by Contrapositive

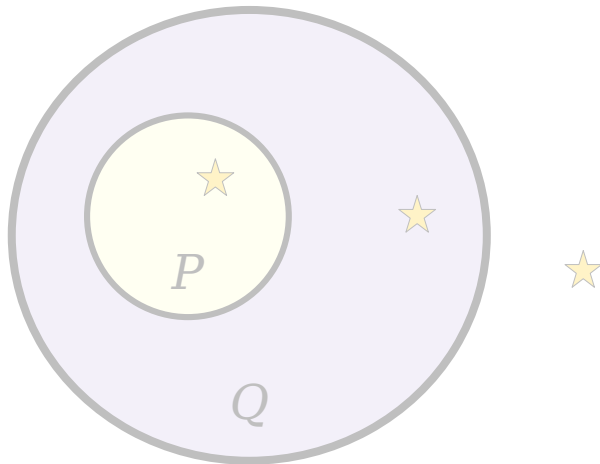
Indirect Proofs: A Story in Four Acts



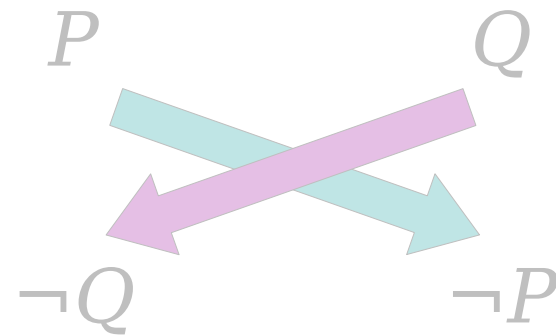
Logical Negation



Proof by Contradiction



Logical Implication



Proof by Contrapositive

Act II

Proof by Contradiction

First, let's reflect on the **direct proof** technique we saw in our previous lecture.

Our First Proof! (from Last Time)

Theorem: If n is an even integer, then n^2 is even.

Our First Proof! (from Last Time)

Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer.

Our First Proof! (from Last Time)

Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Our First Proof! (from Last Time)

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Proof: Assume n is an even integer. We want to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$.

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Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$. This means that

$$n^2 = (2k)^2$$

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Theorem: If n is an even integer, then n^2 is even.

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Since n is even, there is some integer k such that $n = 2k$. This means that

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \end{aligned}$$

Our First Proof! (from Last Time)

Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$. This means that

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

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$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$.

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Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$. This means that

To prove a statement of the form

“If P is true, then Q is true,”

start by asking the reader to
assume that **P** is true.

From this, we see
($2k^2$) where $n^2 = 2k^2$
what we wanted

Our First Proof! (from Last Time)

Theorem: If n is an even integer, then n^2 is even.

Proof: Assume n is an even integer. We want to show that n^2 is even.

Since n is even, there is some integer k such that $n = 2k$. This means that

If we apply sound logic (using definitions, algebra, etc.) all the statements that follow are also true.

From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

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that n^2 is even.

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From this, we see that there is an integer m (namely, $2k^2$) where $n^2 = 2m$. Therefore, n^2 is even, which is what we wanted to show. ■

More generally speaking,
the process looks like this:

Direct Proof

We start with a statement (or statements) we know (or assume) to be true.

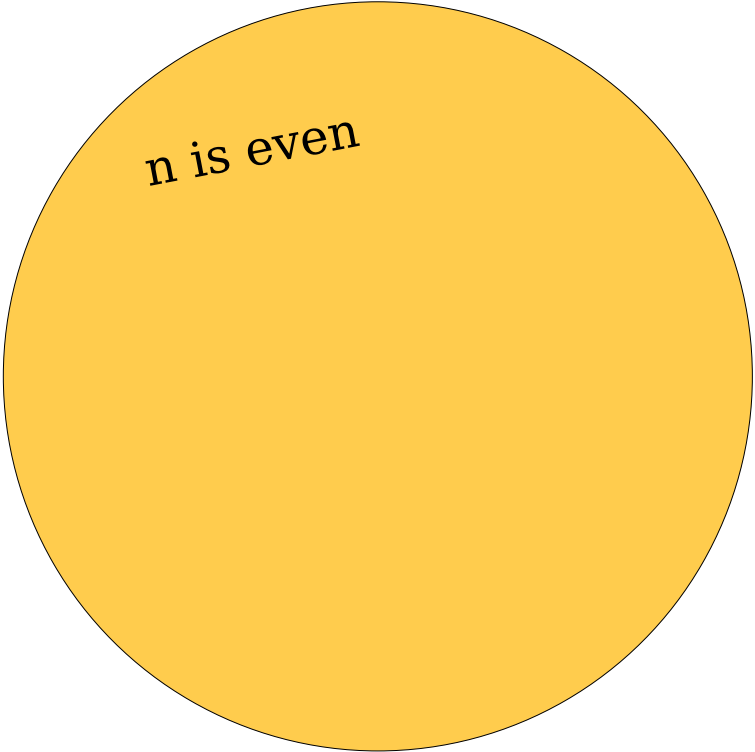
Direct Proof

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n is even

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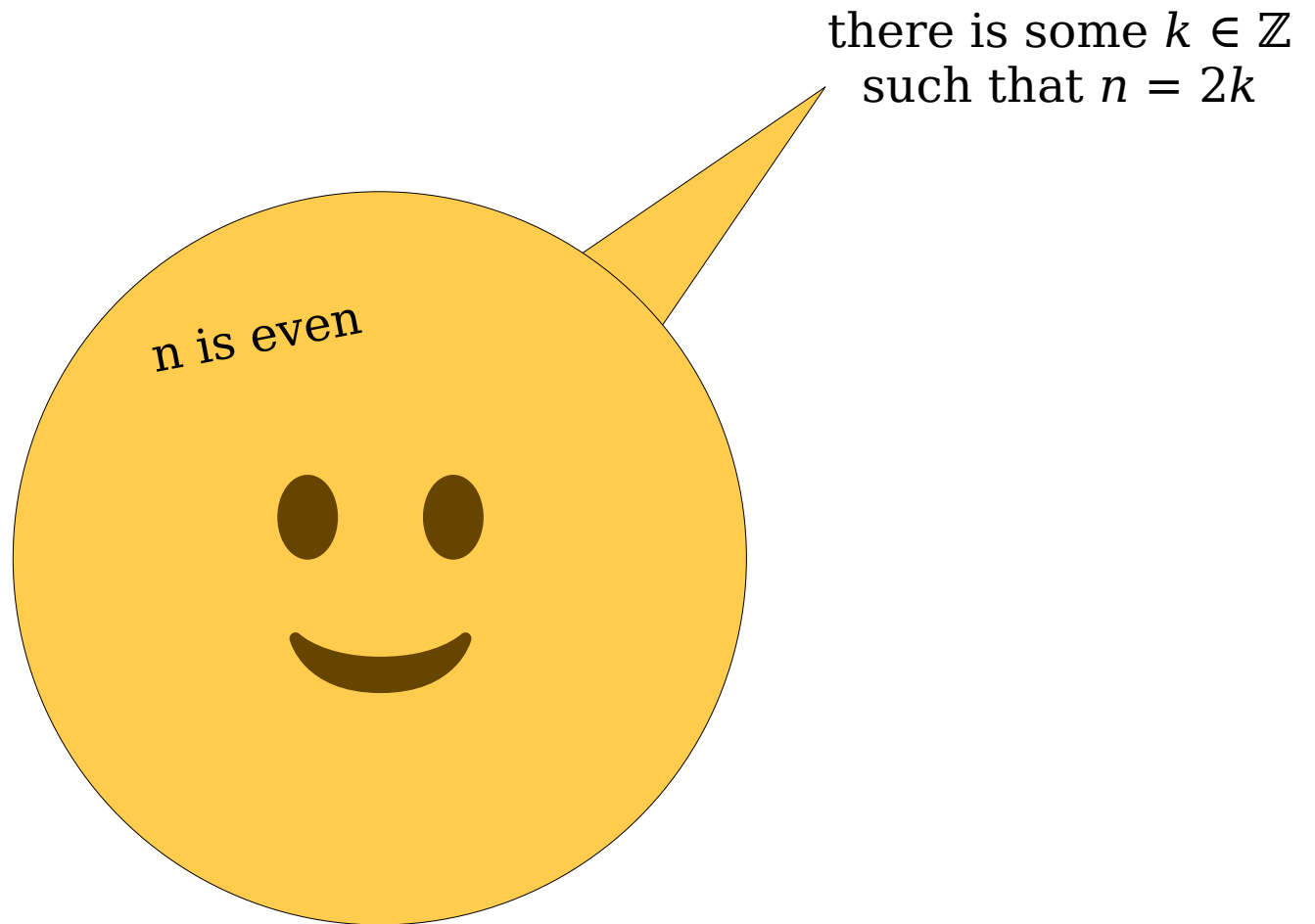


Direct Proof

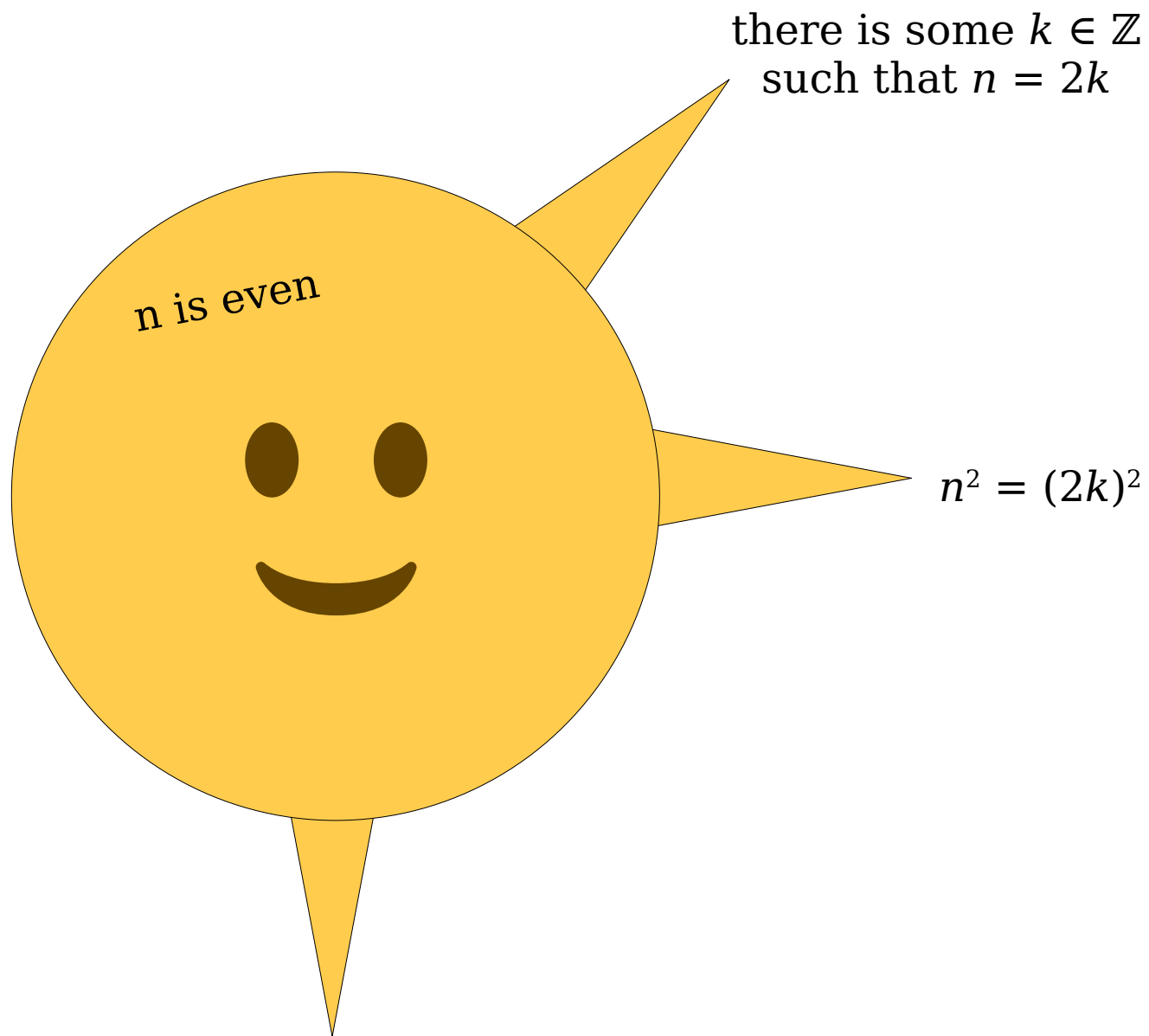


Next, we apply sound logic and rational argument to arrive at other true statements!

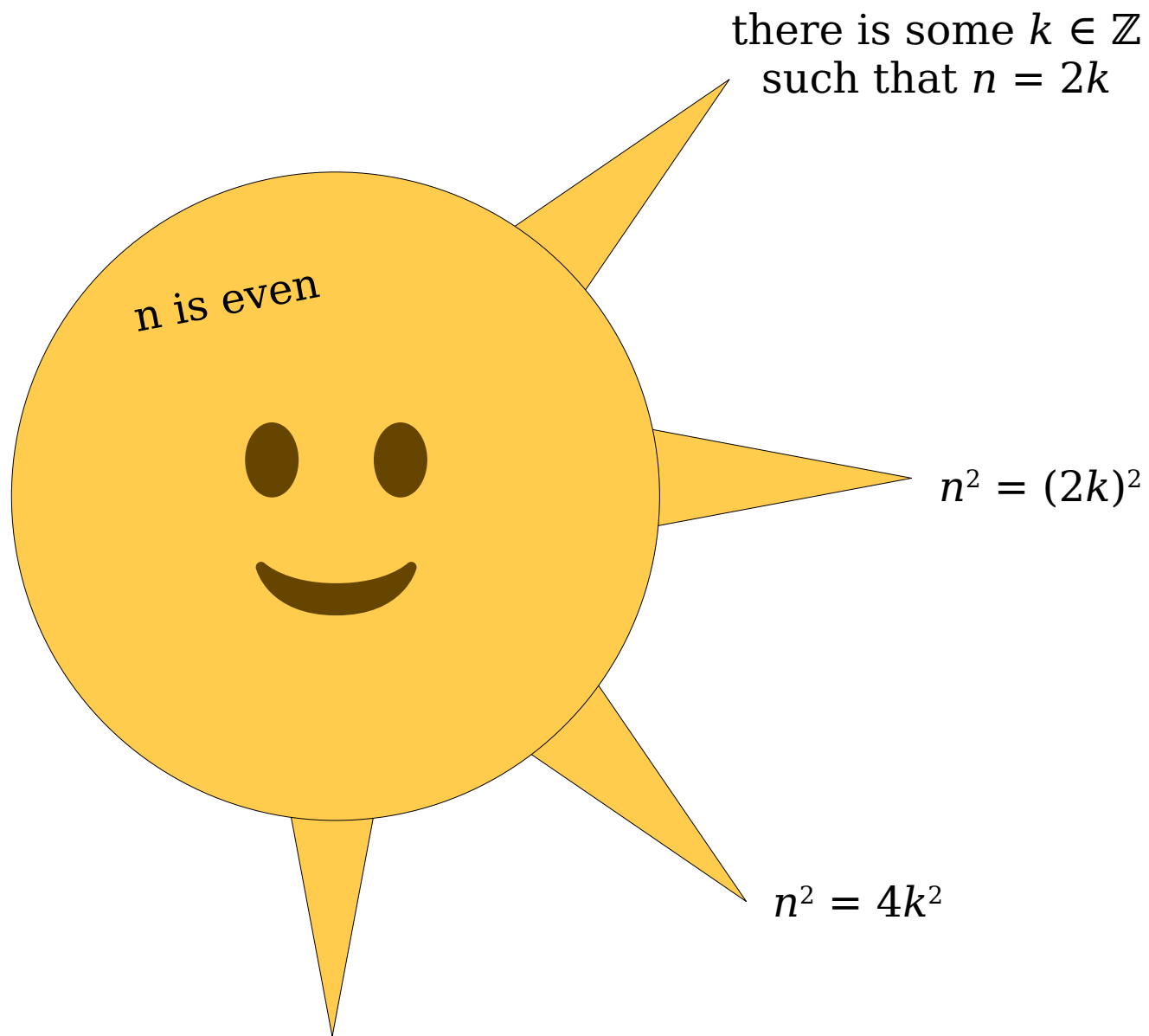
Direct Proof



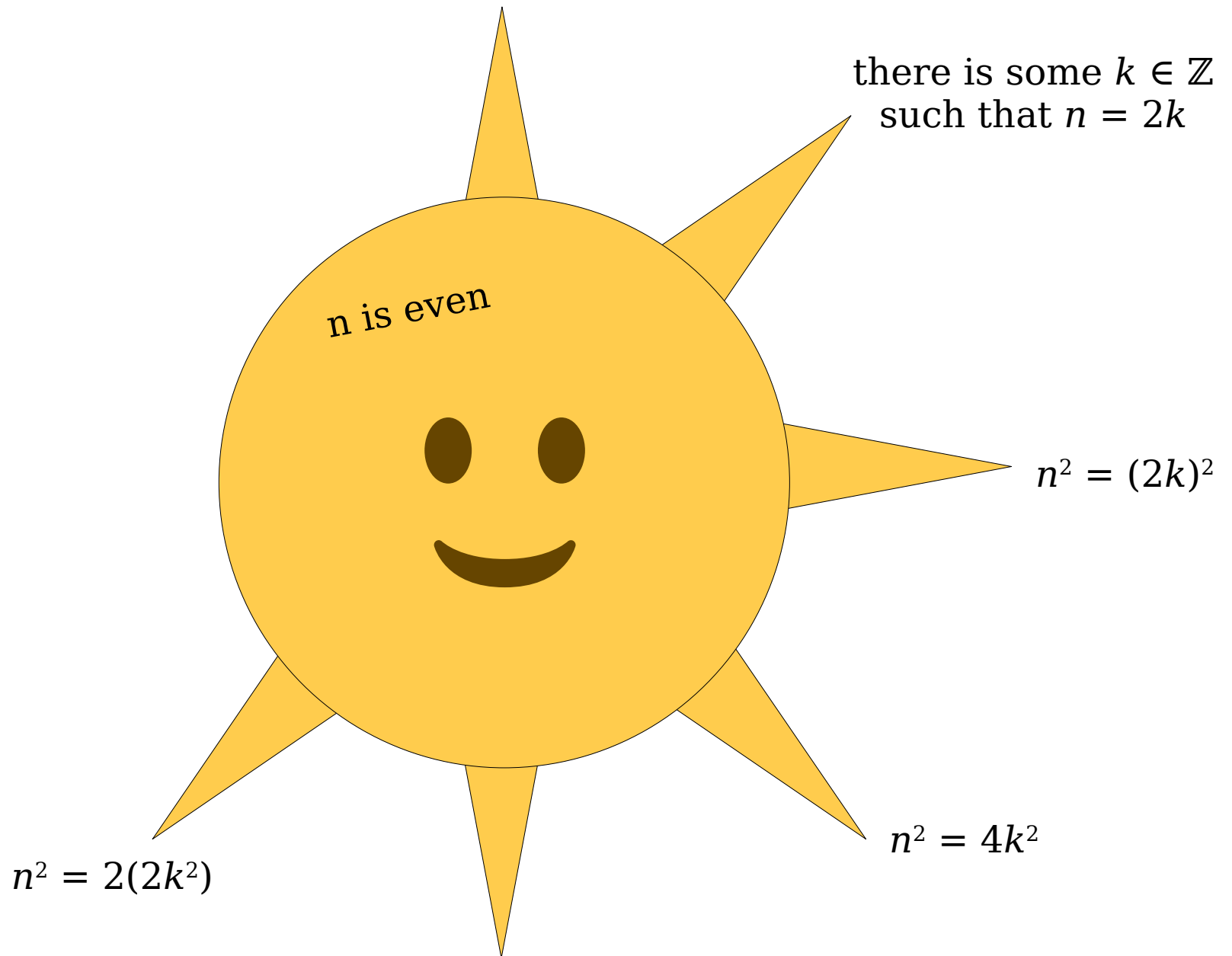
Direct Proof



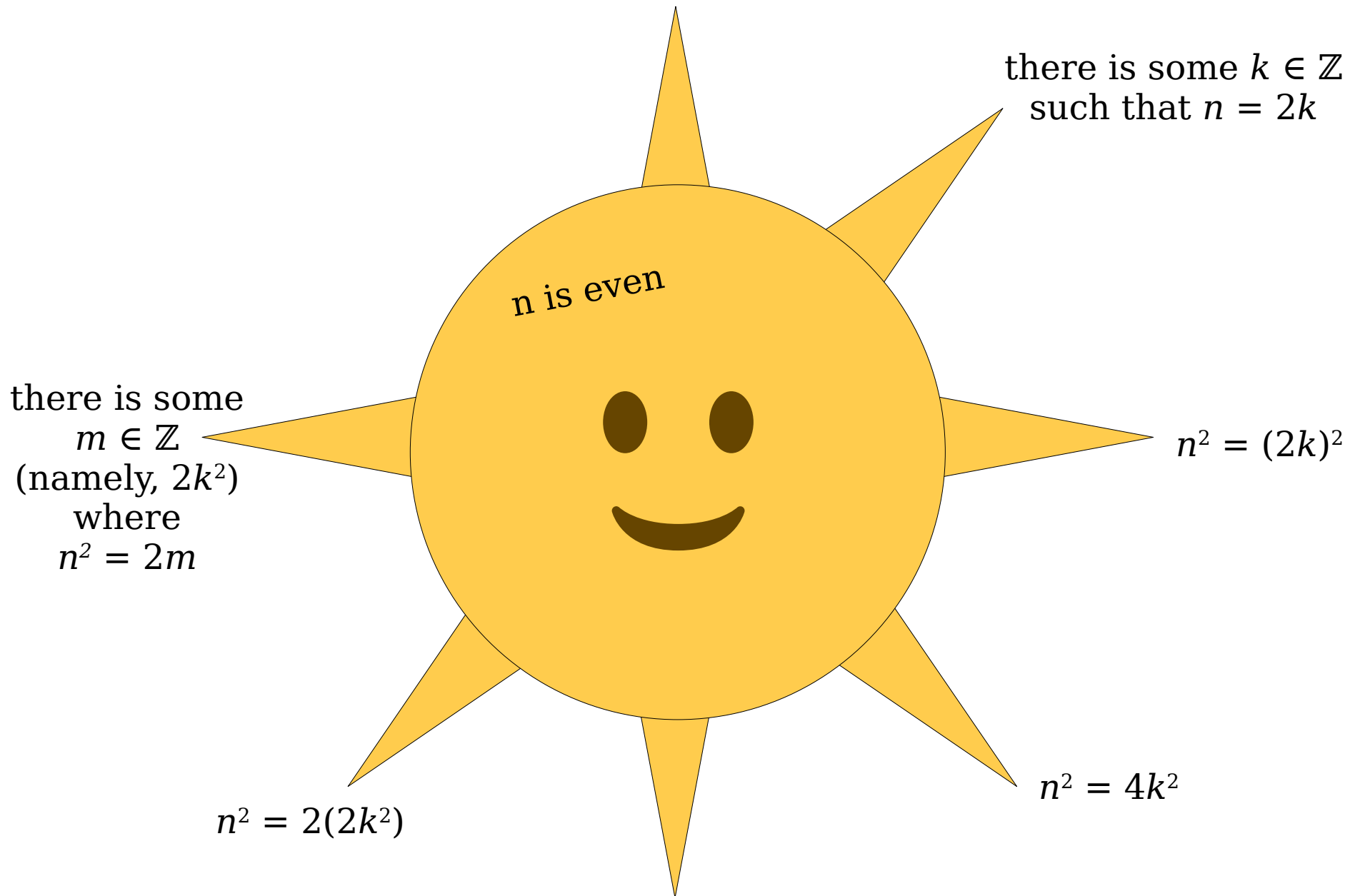
Direct Proof



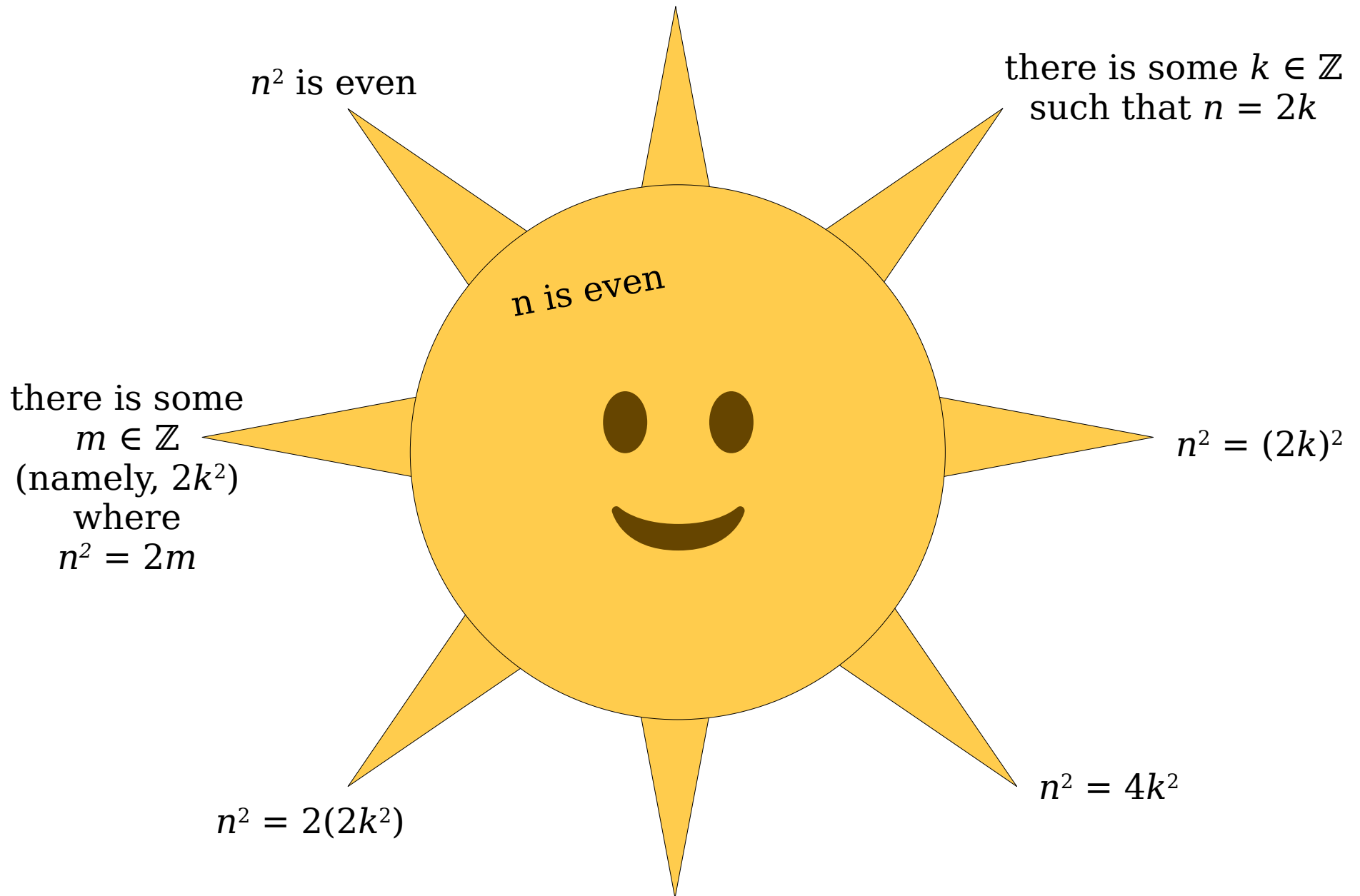
Direct Proof



Direct Proof



Direct Proof



Direct Proof



n^2 is even

there is some $k \in \mathbb{Z}$
such that $n = 2k$

n is even

there is some
 $m \in \mathbb{Z}$
(namely, $2k^2$)
where
 $n^2 = 2m$

$$n^2 = (2k)^2$$

$$n^2 = 2(2k^2)$$

$$n^2 = 4k^2$$

Direct Proof



n^2 is even



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n is even

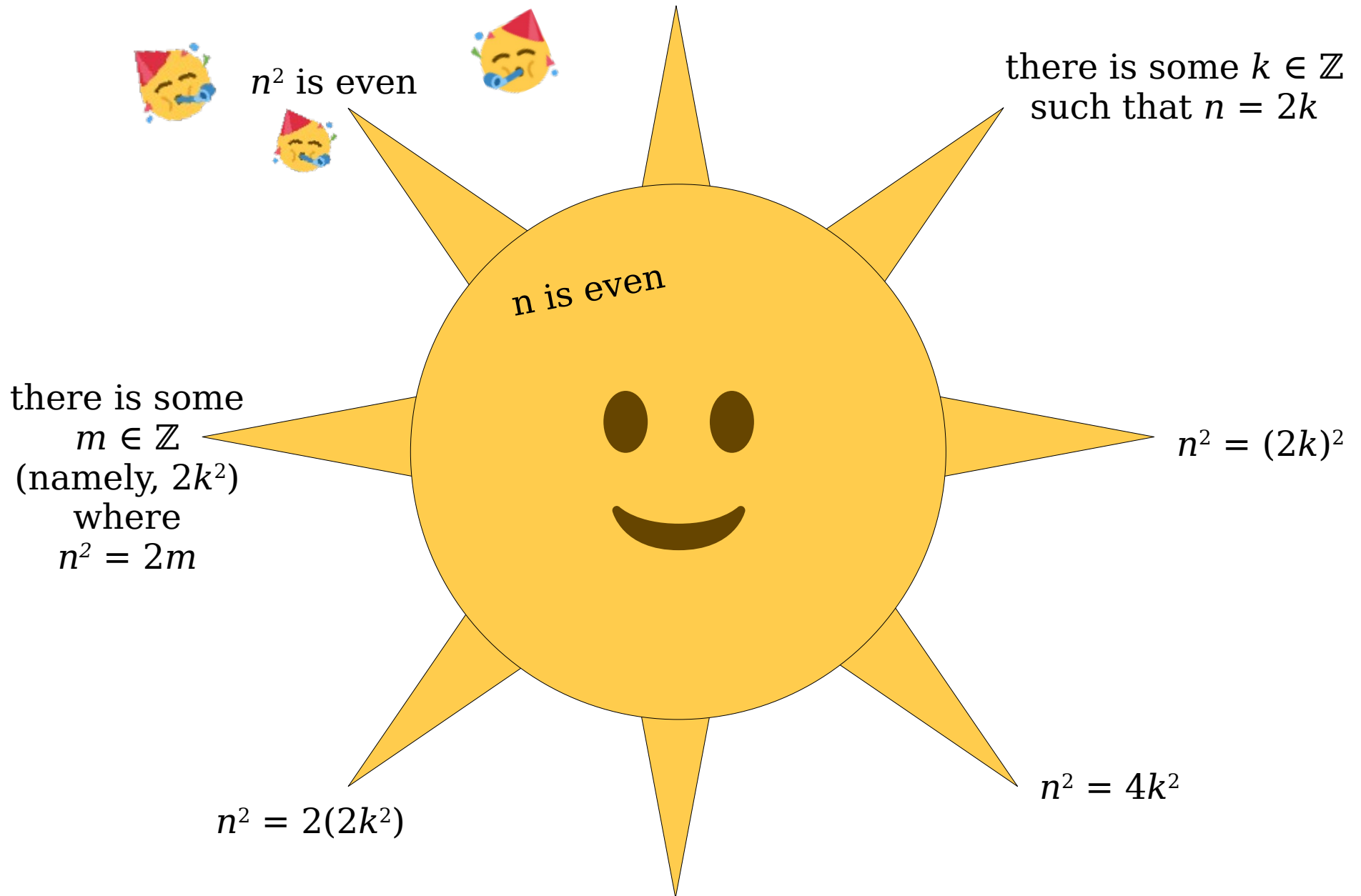
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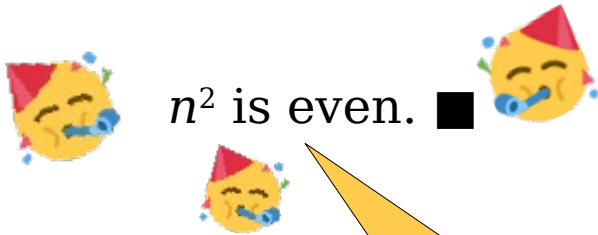
$$n^2 = 2(2k^2)$$

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Direct Proof



Direct Proof



n^2 is even. ■

there is some $k \in \mathbb{Z}$
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n is even

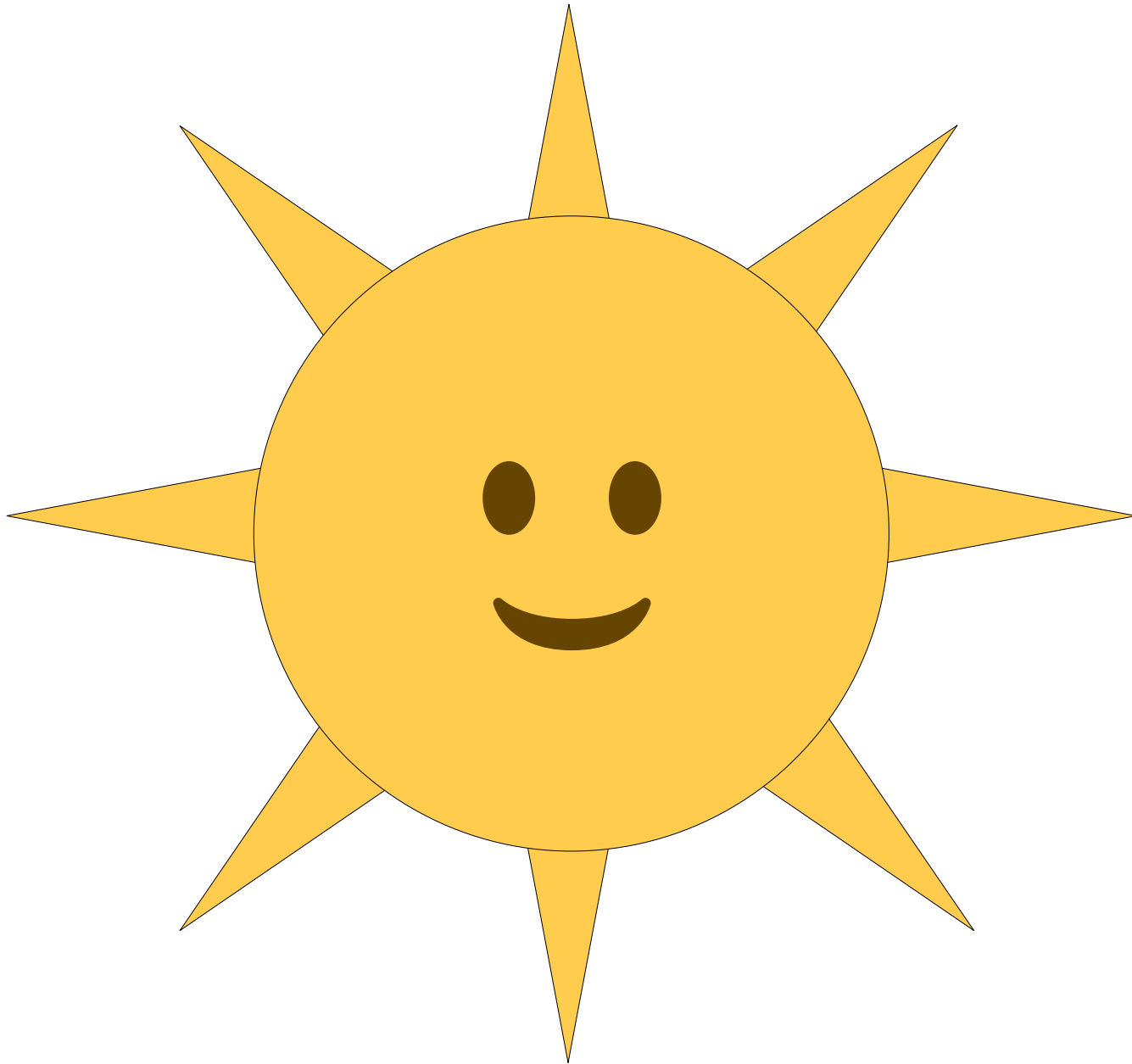
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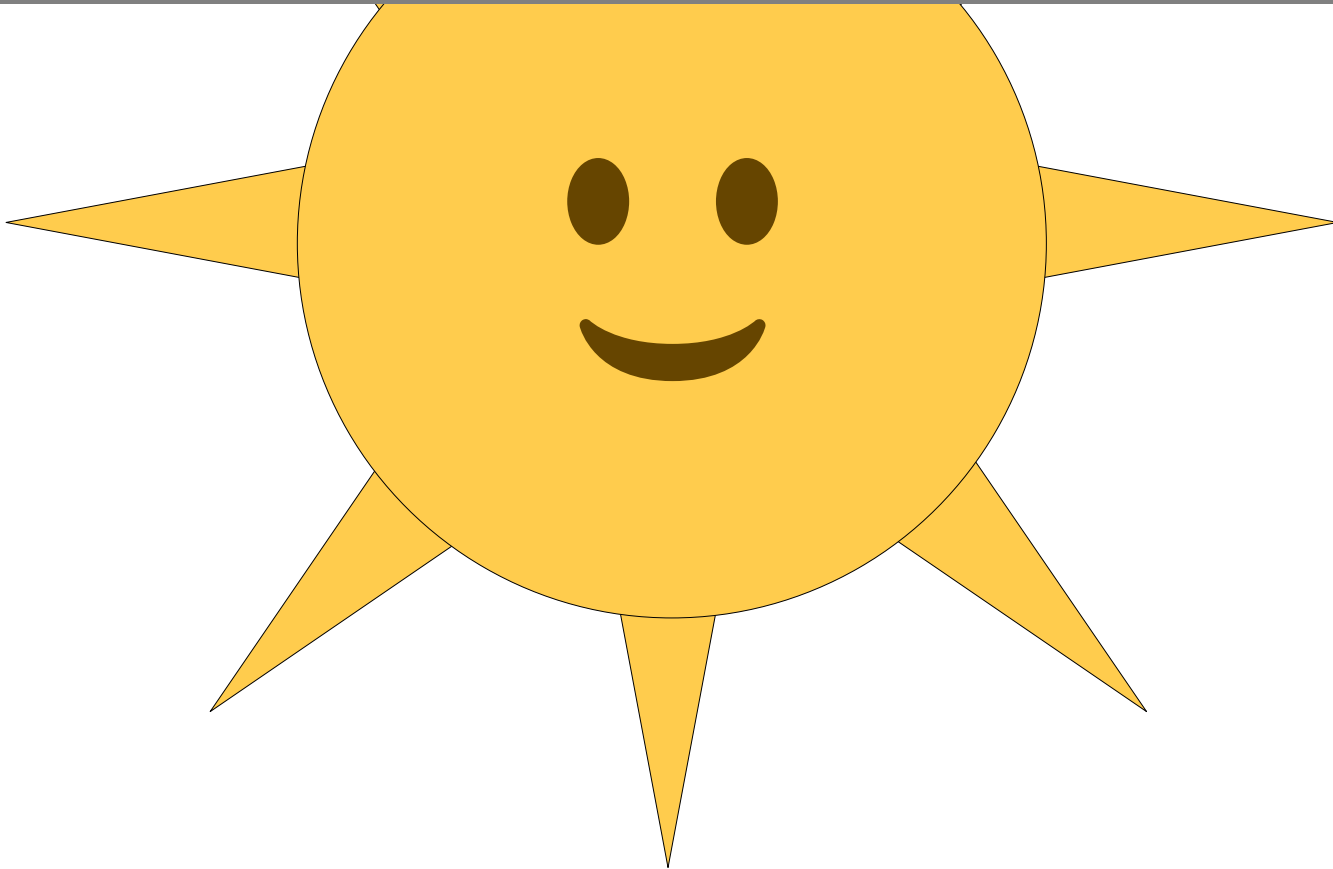
$$n^2 = 4k^2$$

Direct Proof



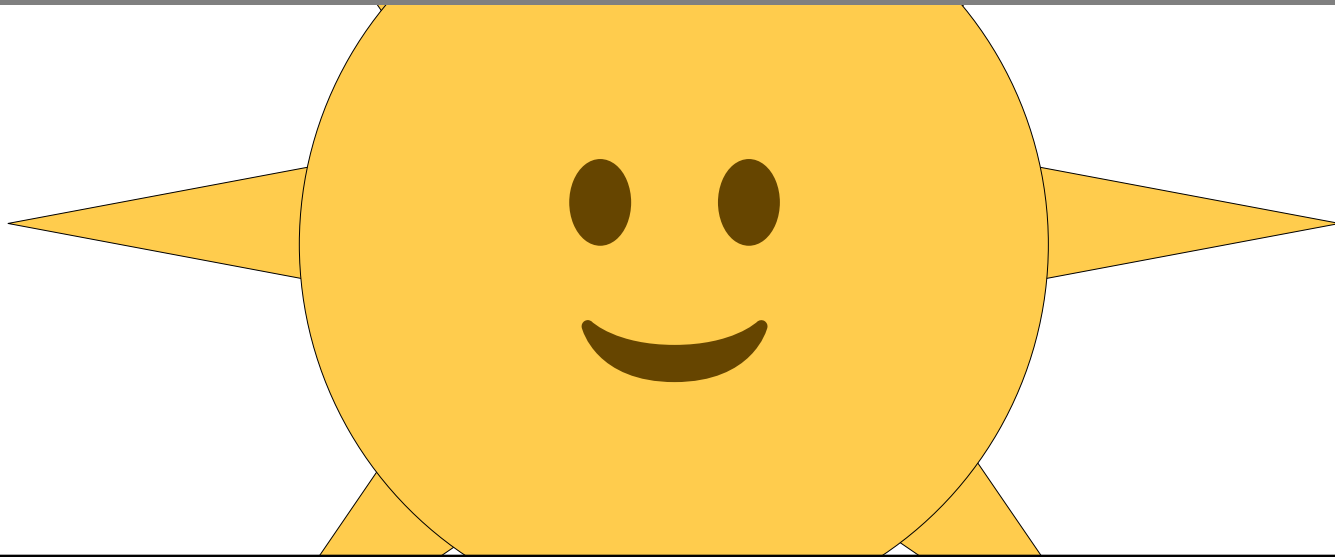
Direct Proof

Key Takeaway: When we apply sound logic
to true statements...



Direct Proof

Key Takeaway: When we apply sound logic
to true statements...



the process radiates truth with the power
and intensity of a thousand burning suns!

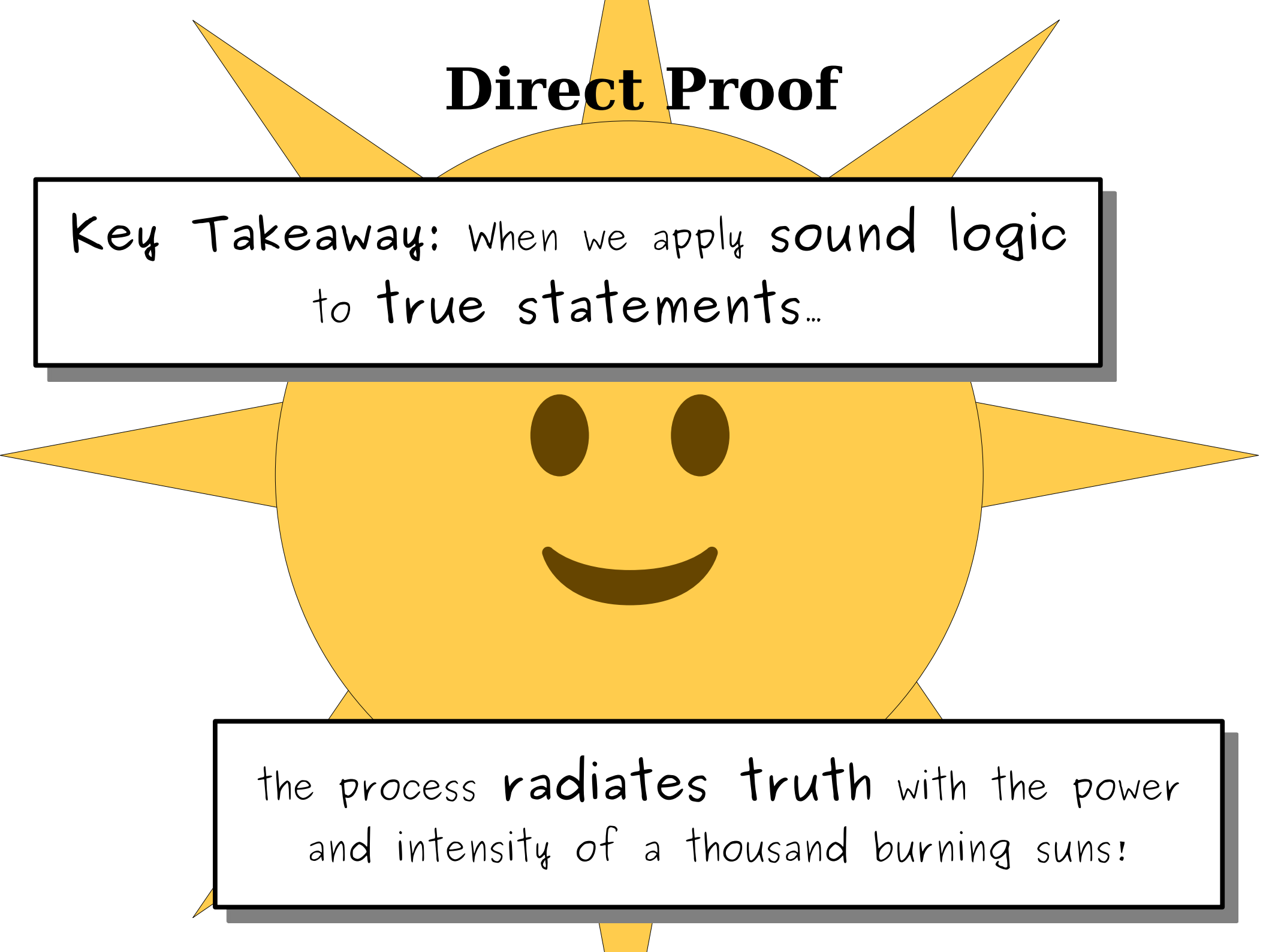
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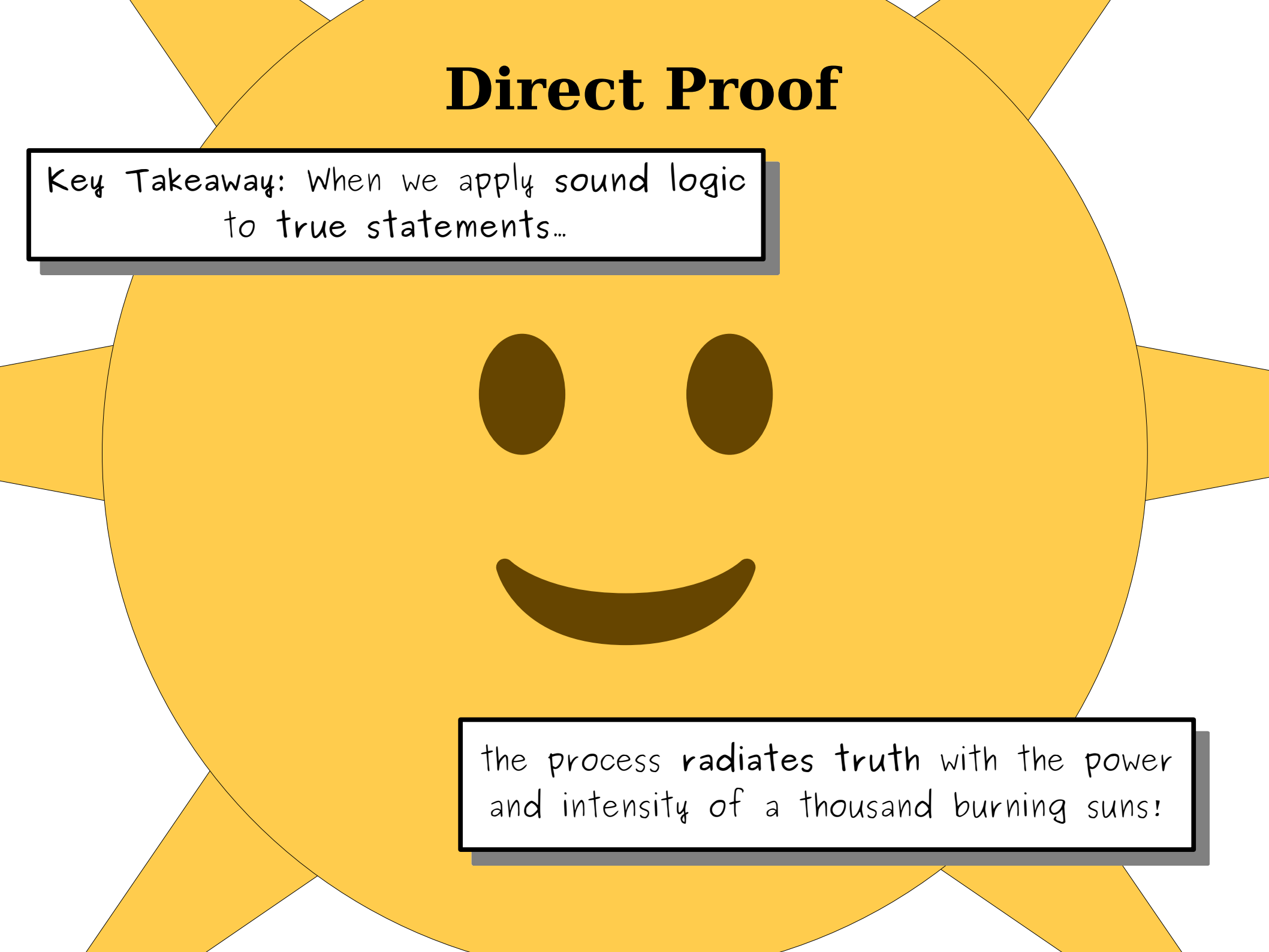
Direct Proof



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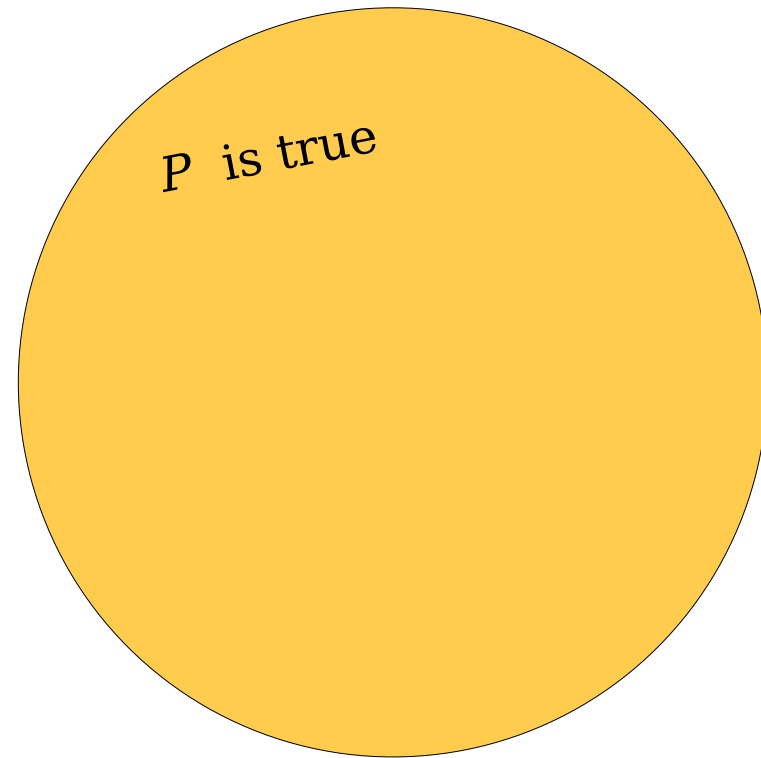
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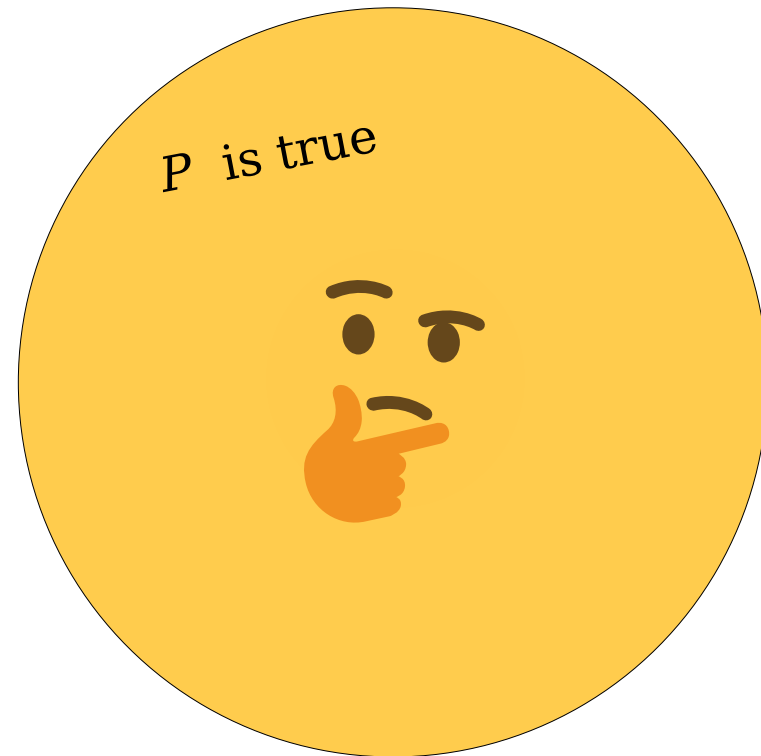


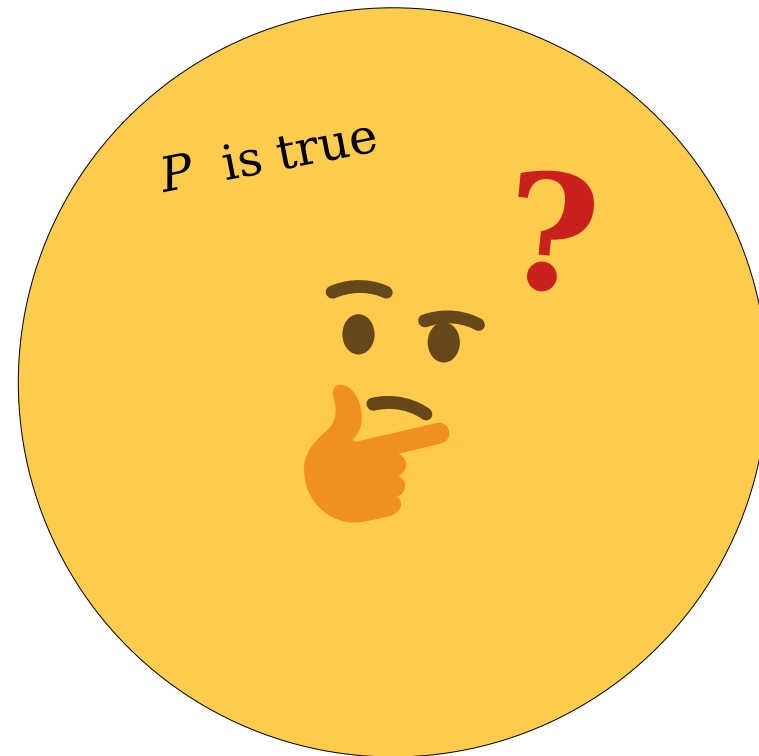
Okay, but...

Okay, but...

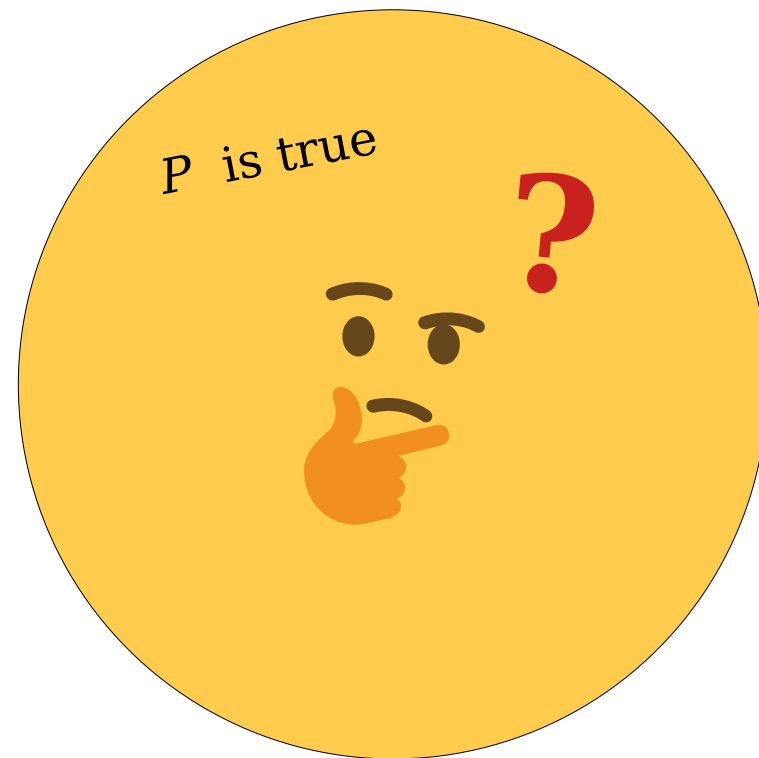
what if we start with a proposition
whose **truthiness** is **unknown** to us?



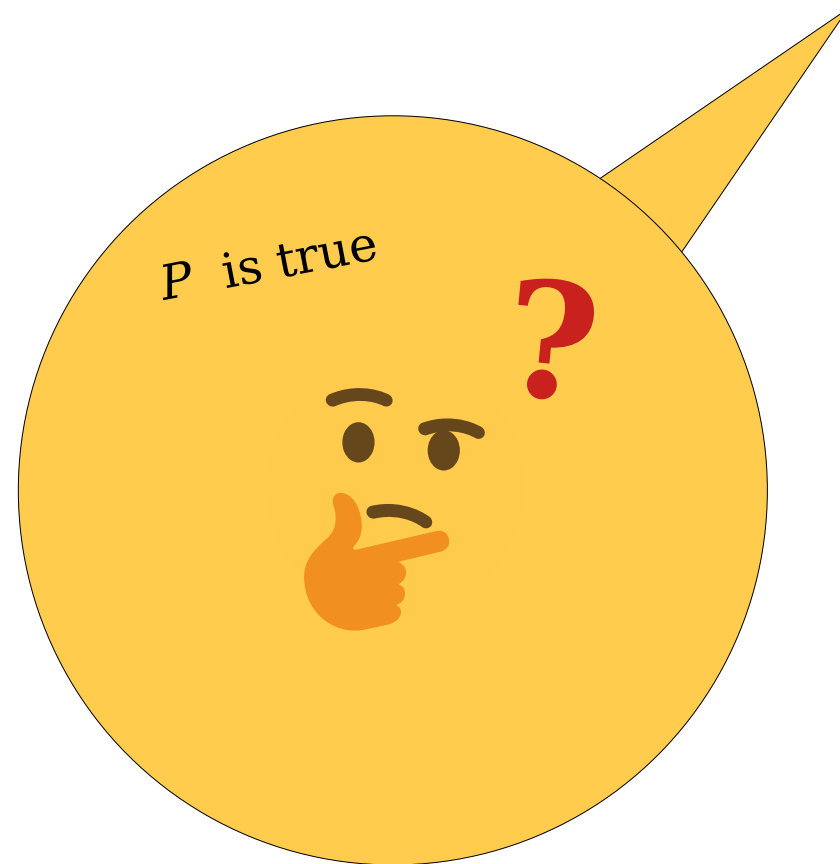




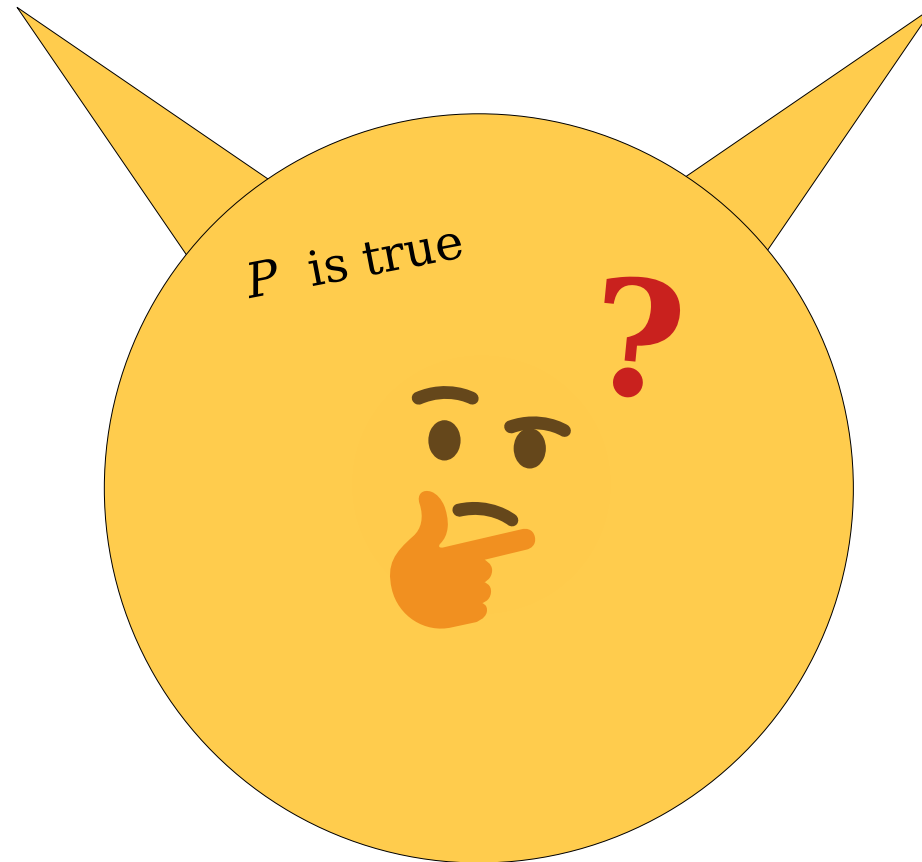
This might radiate
SOME truth.



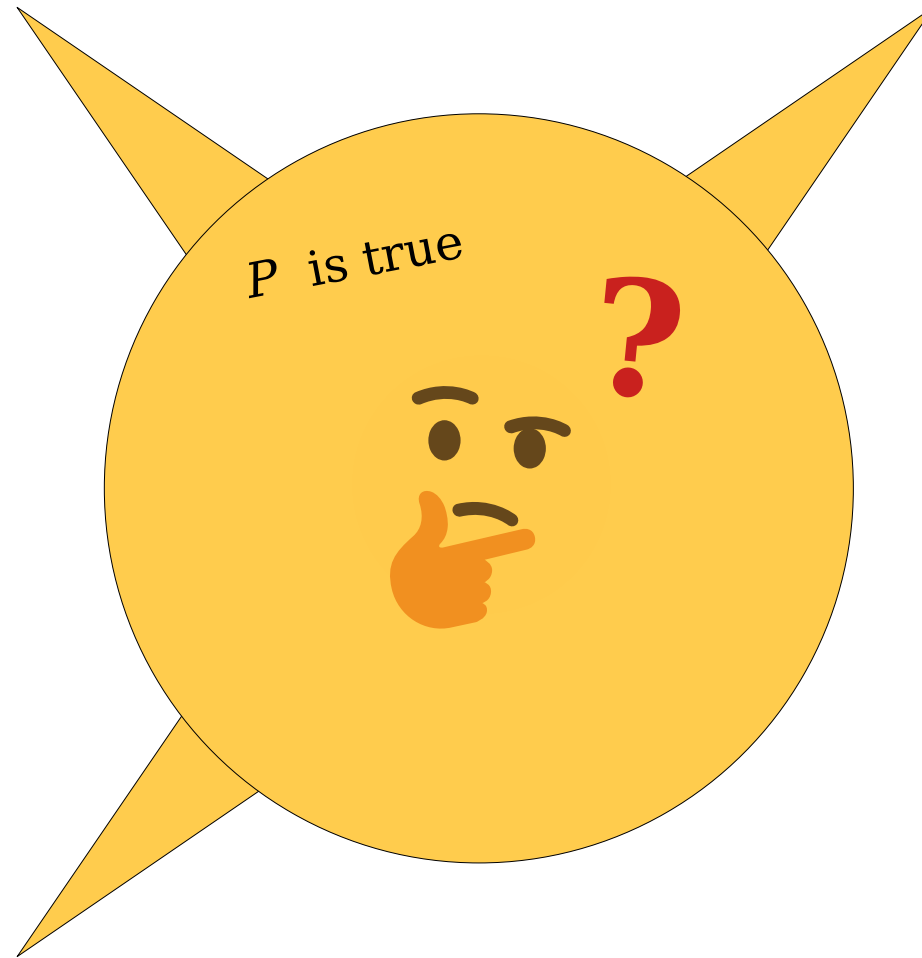
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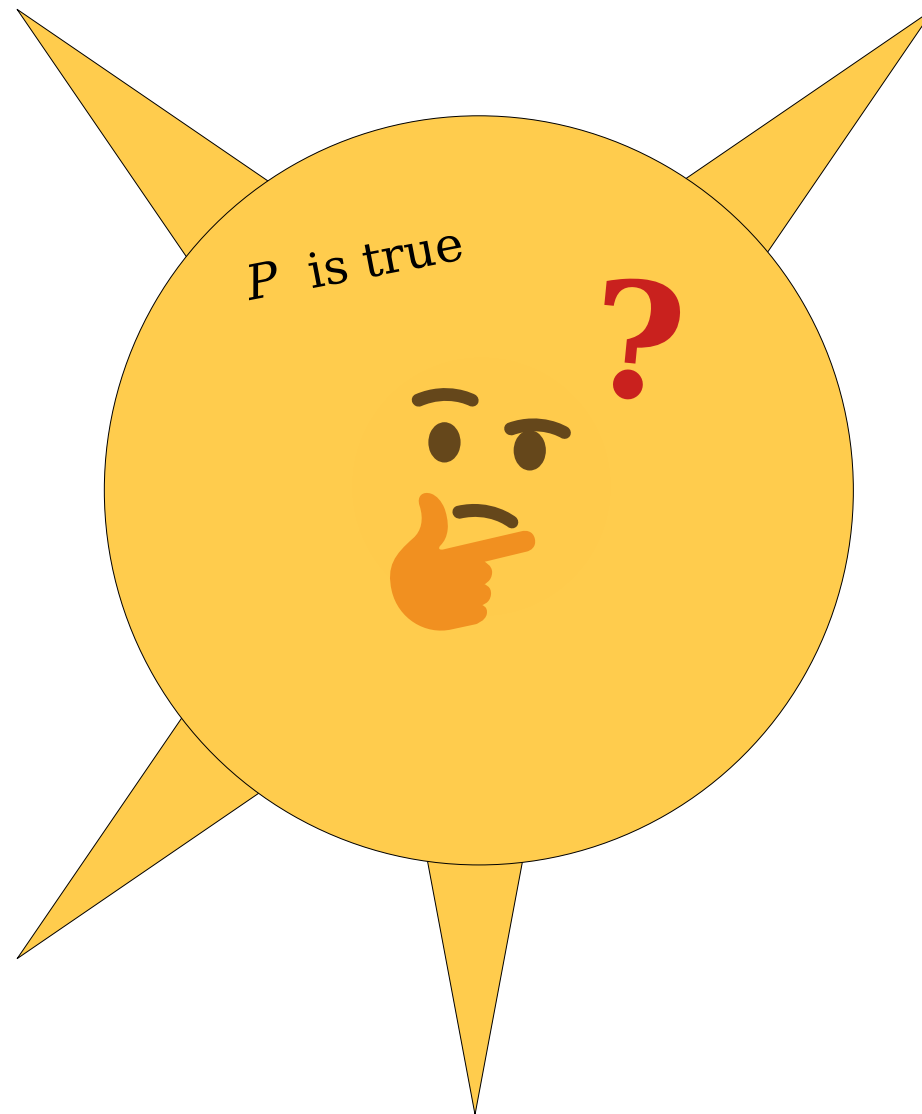
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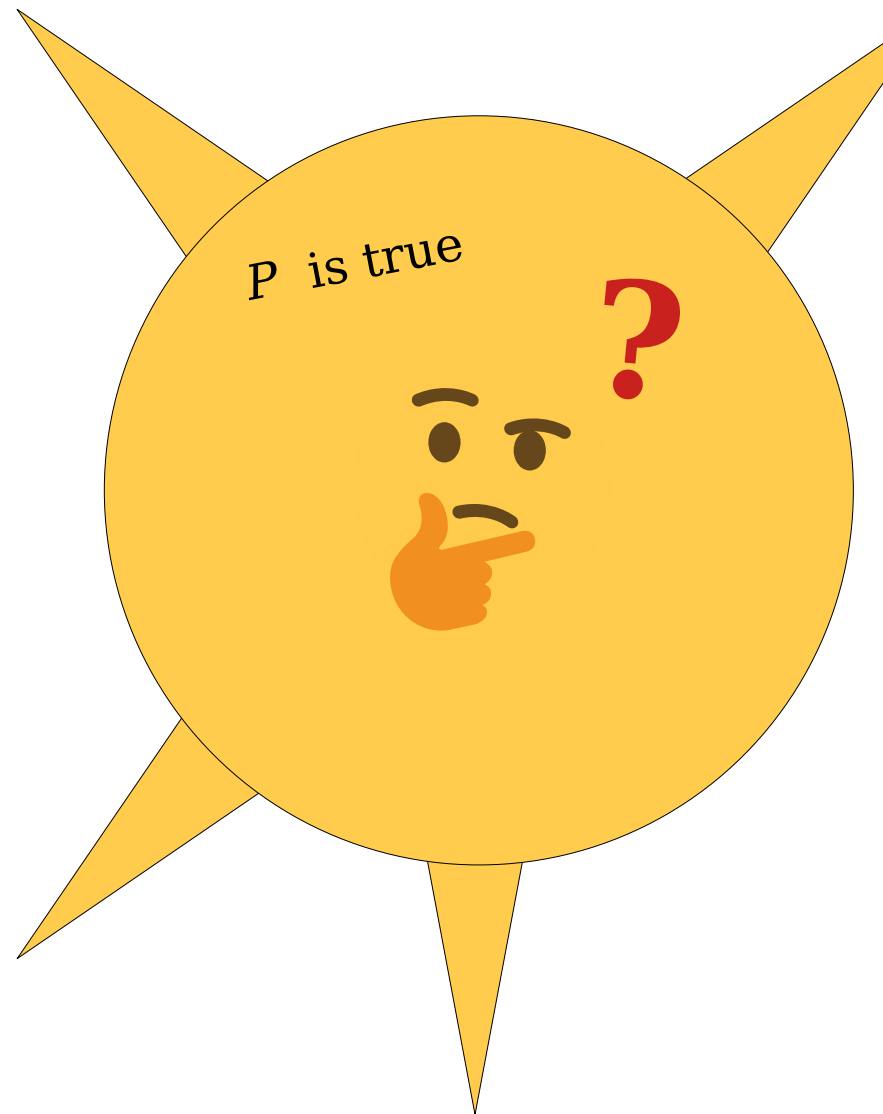


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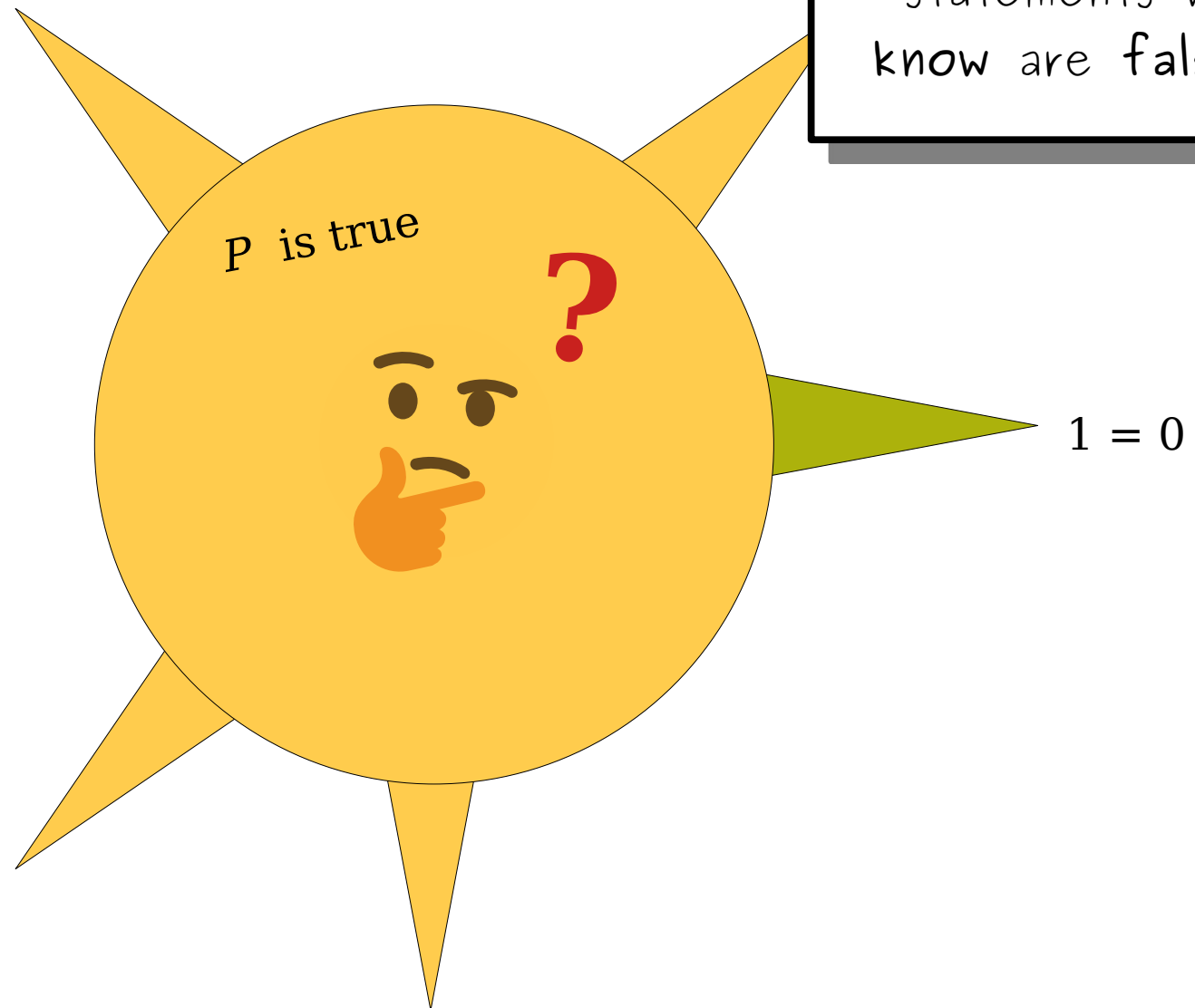
This might radiate
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What does it mean
if we start
generating
statements we
know are false?



This might radiate
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What does it mean
if we start
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know are false?



$1 = 0$

This might radiate
SOME truth.

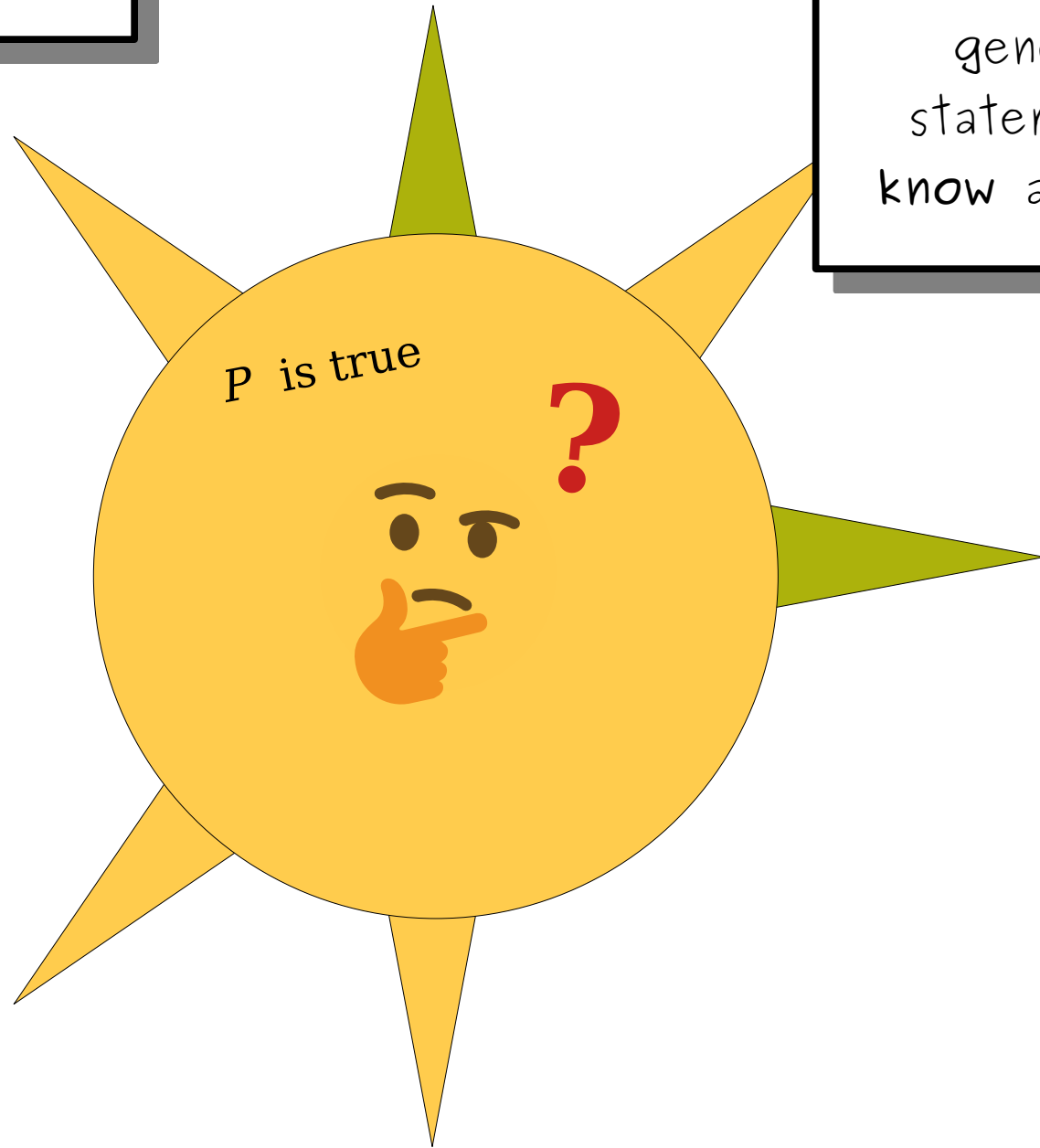
3 is even

What does it mean
if we start
generating
statements we
know are false?

P is true

?

$1 = 0$



This might radiate
SOME truth.

3 is even

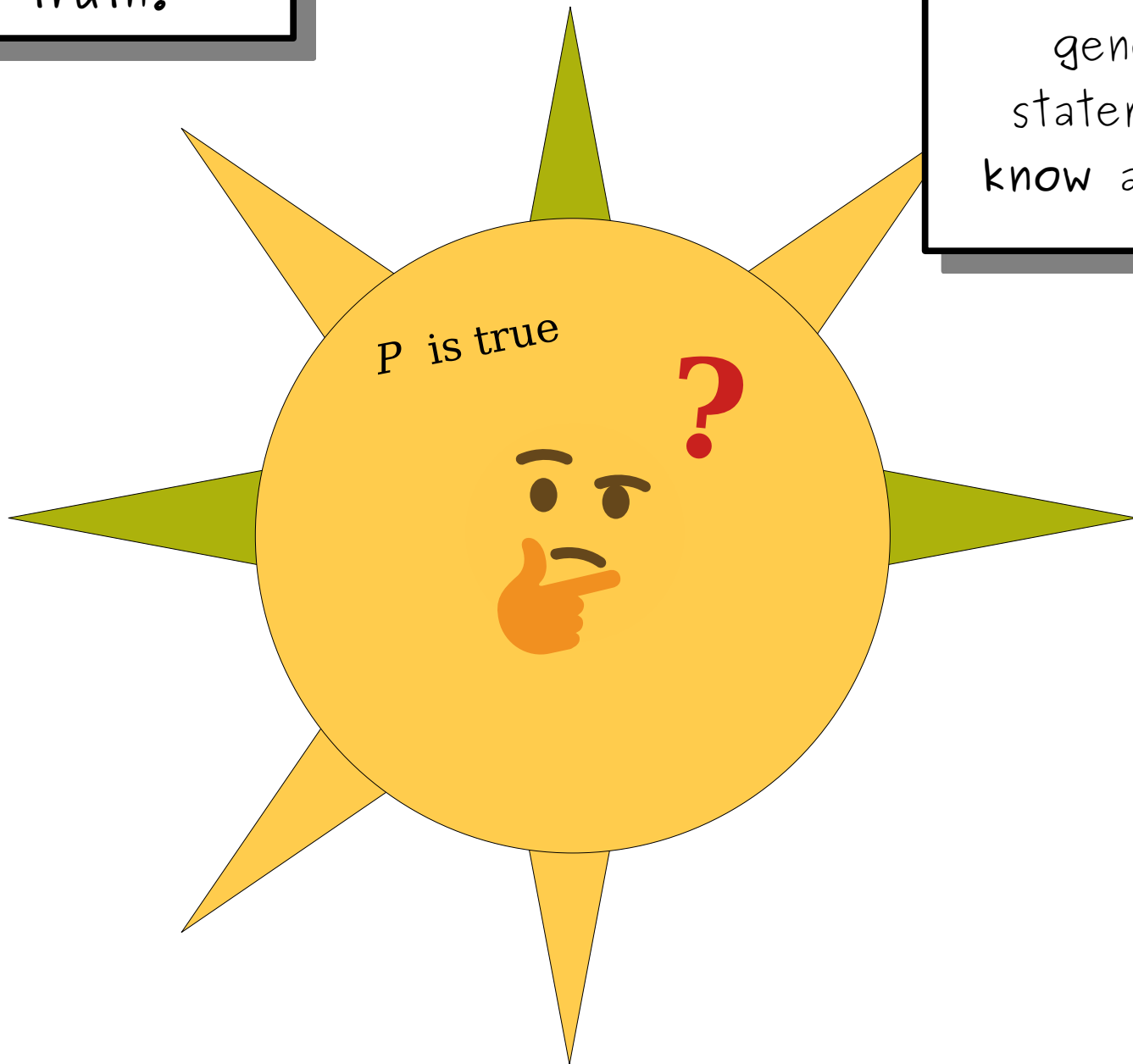
What does it mean
if we start
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P is true

?

$5 > 10$

$1 = 0$



This might radiate
SOME truth.

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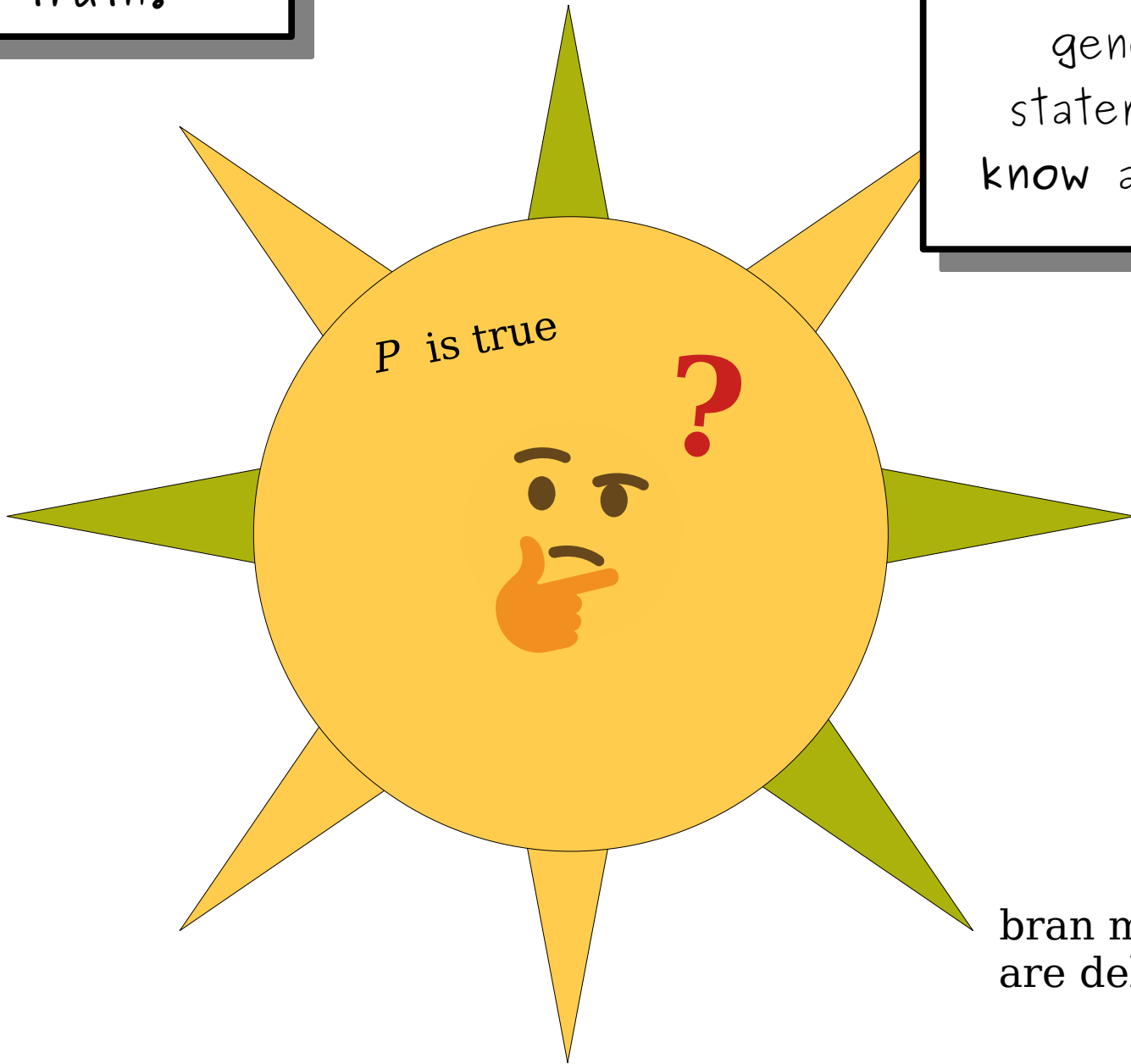
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bran muffins
are delicious



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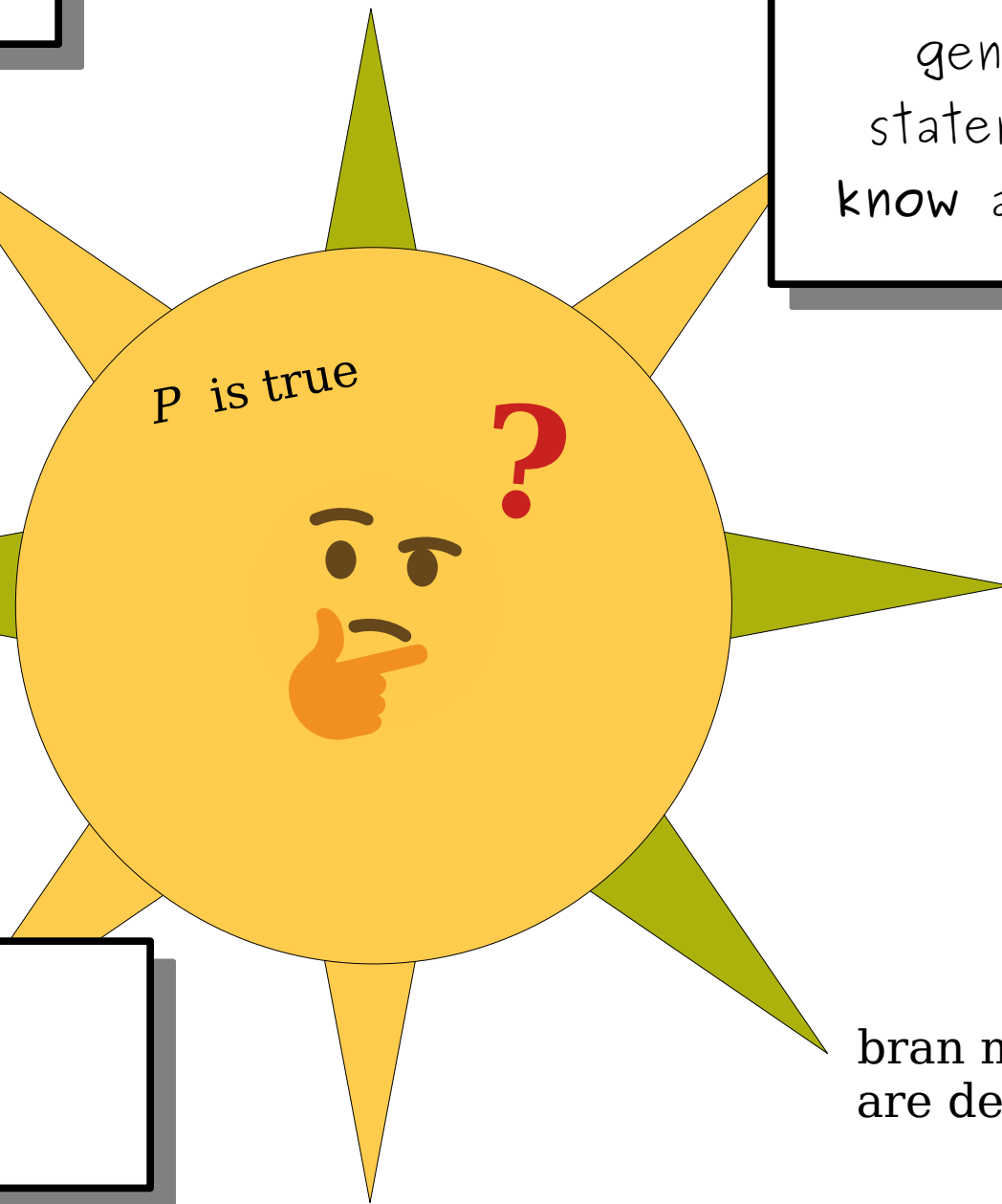
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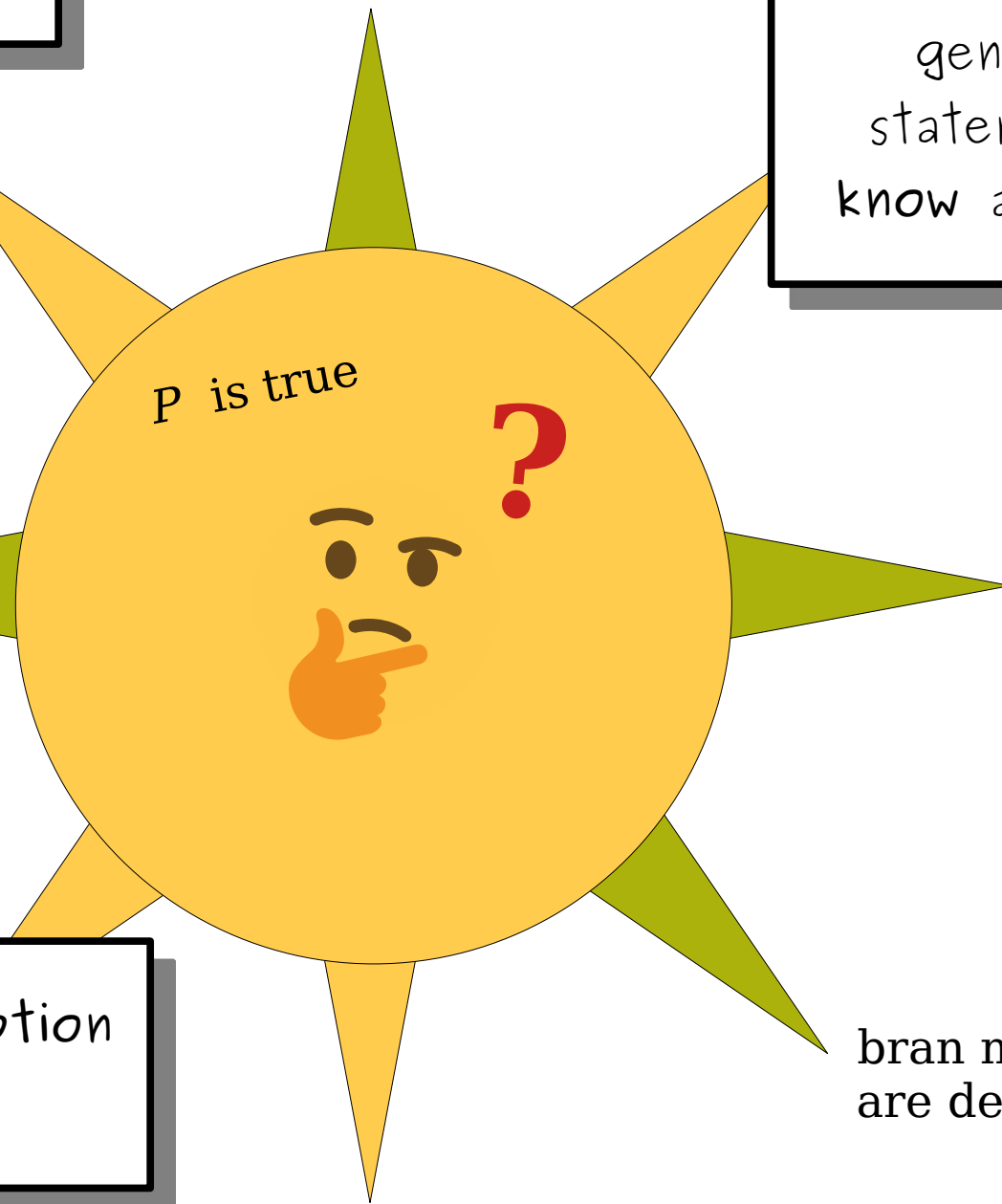
?

$5 > 10$

$1 = 0$

Our original assumption
must be false!

bran muffins
are delicious



This might radiate
SOME truth.

3 is even

What does it mean
if we start
generating
statements we
know are false?

~~P is true~~

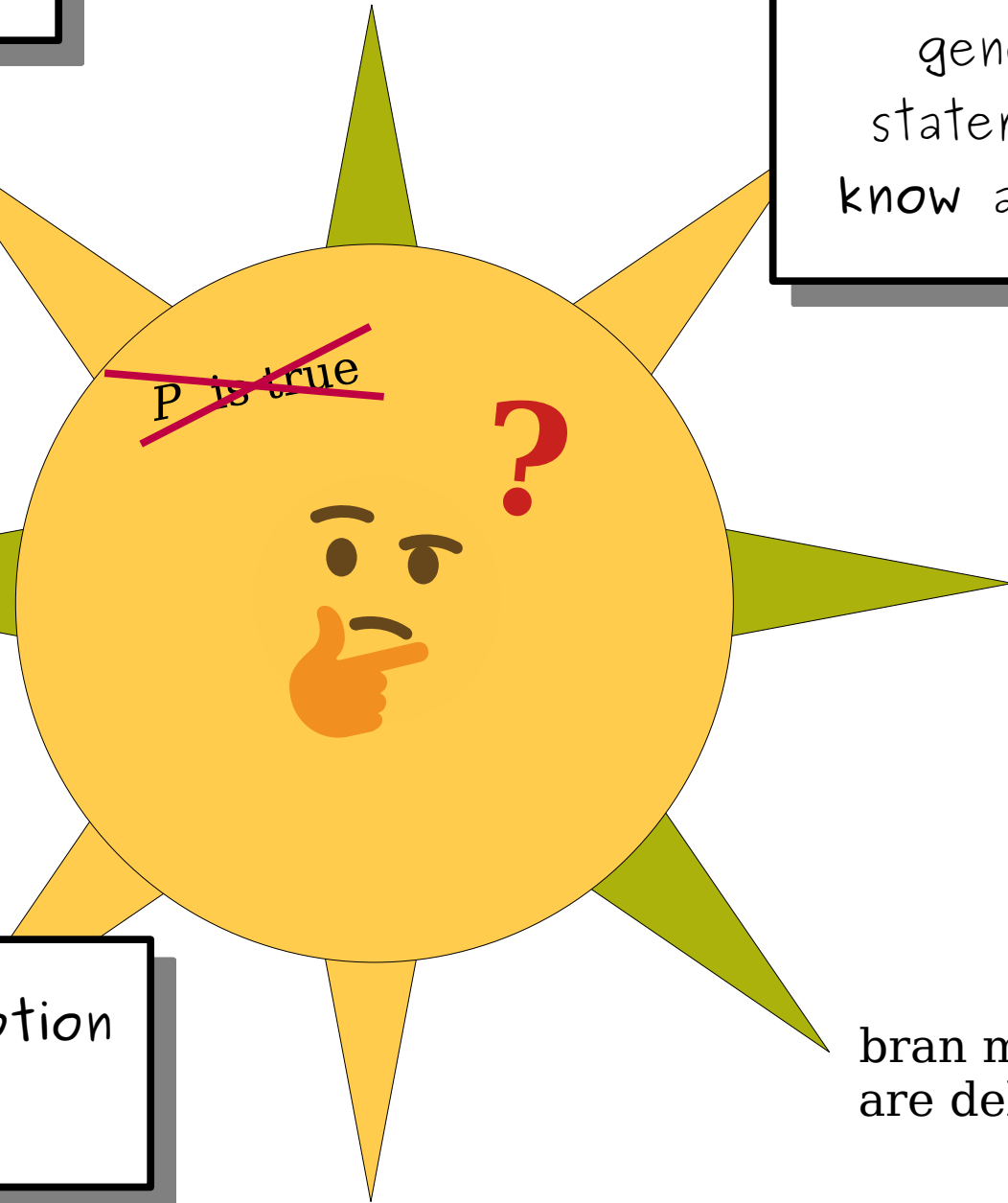
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$5 > 10$

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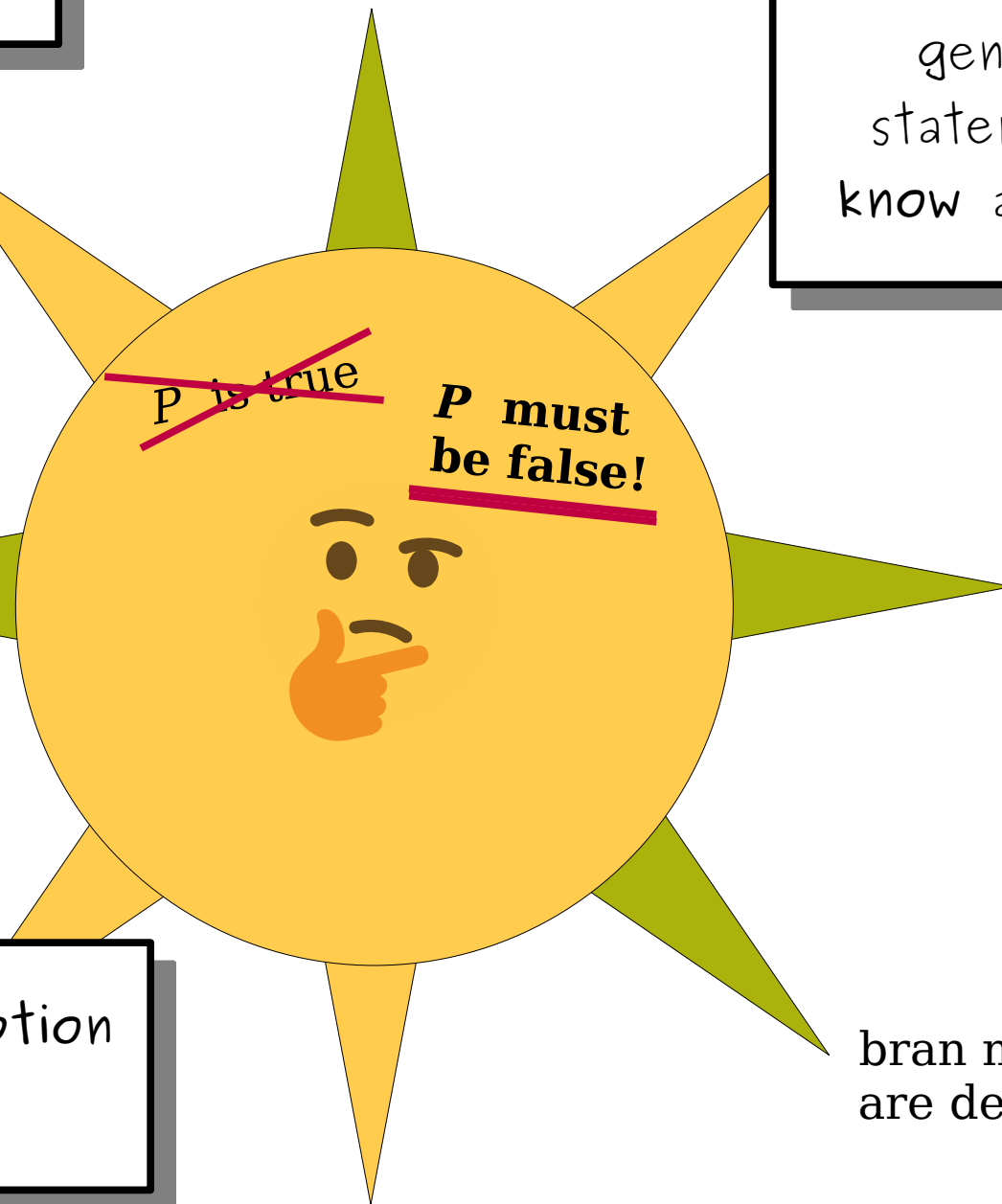
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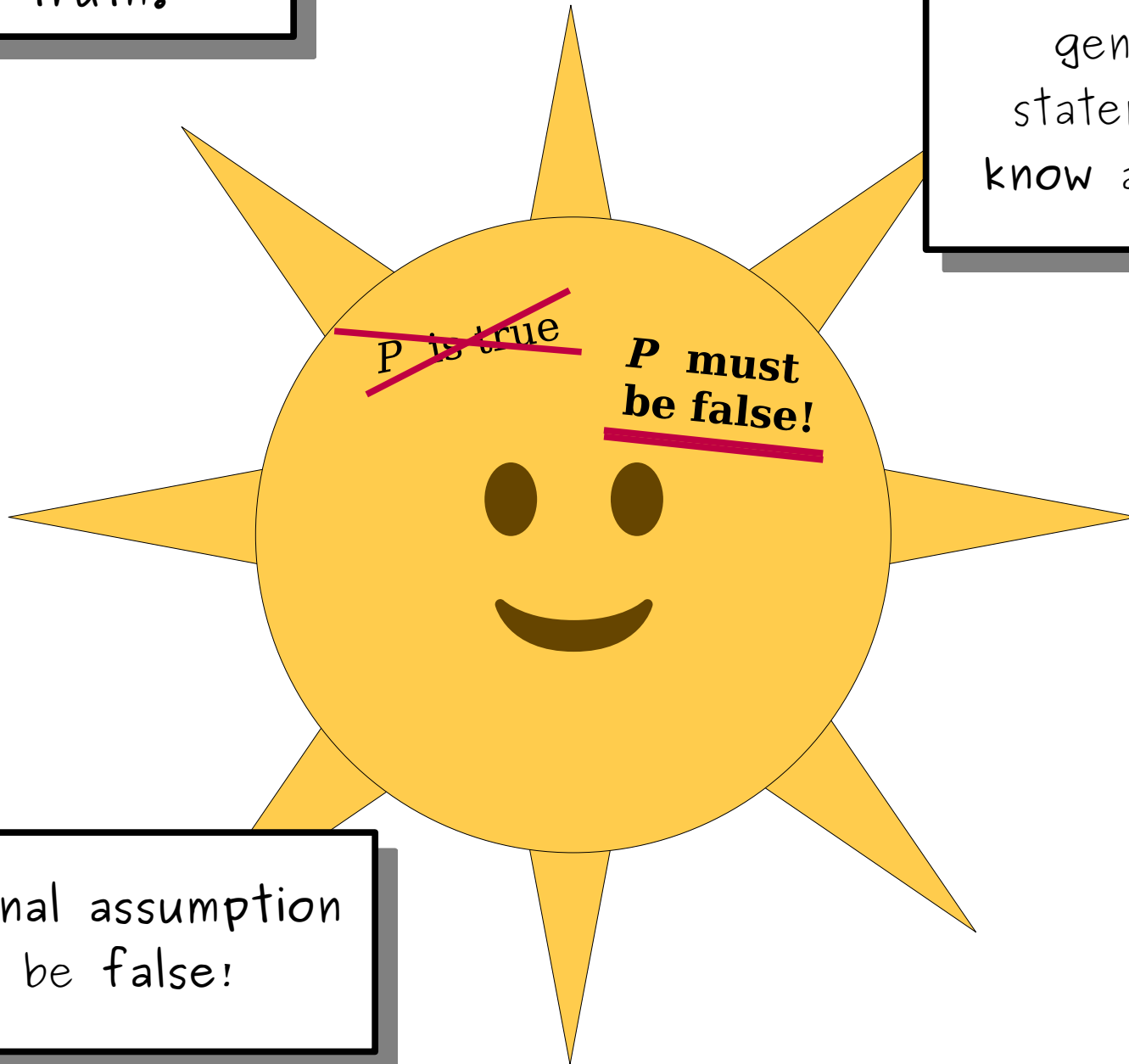
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This gives rise to a powerful proof technique called **proof by contradiction**!

Suppose we want to use
this technique to show
that P is true.



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What proposition can we place
into the Zone of Uncertainty
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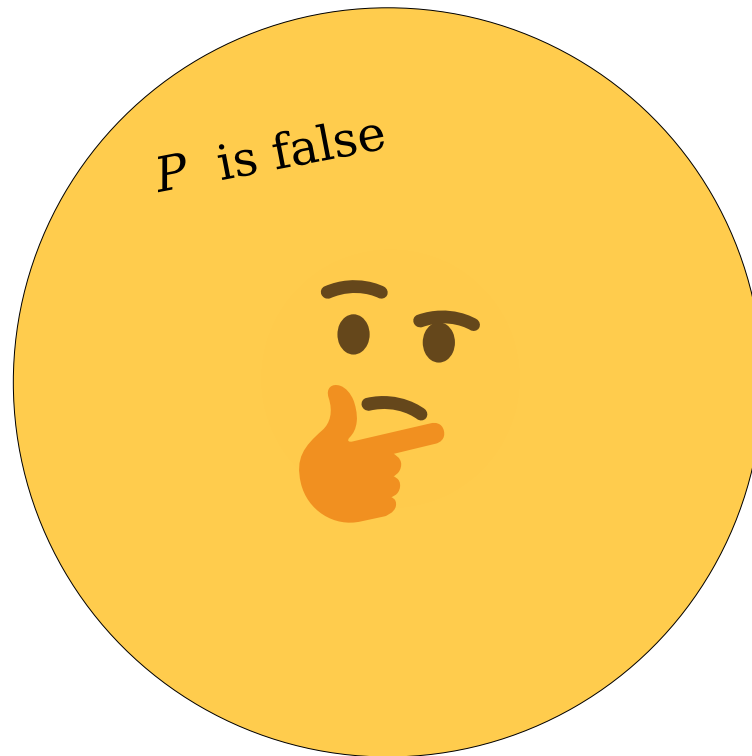
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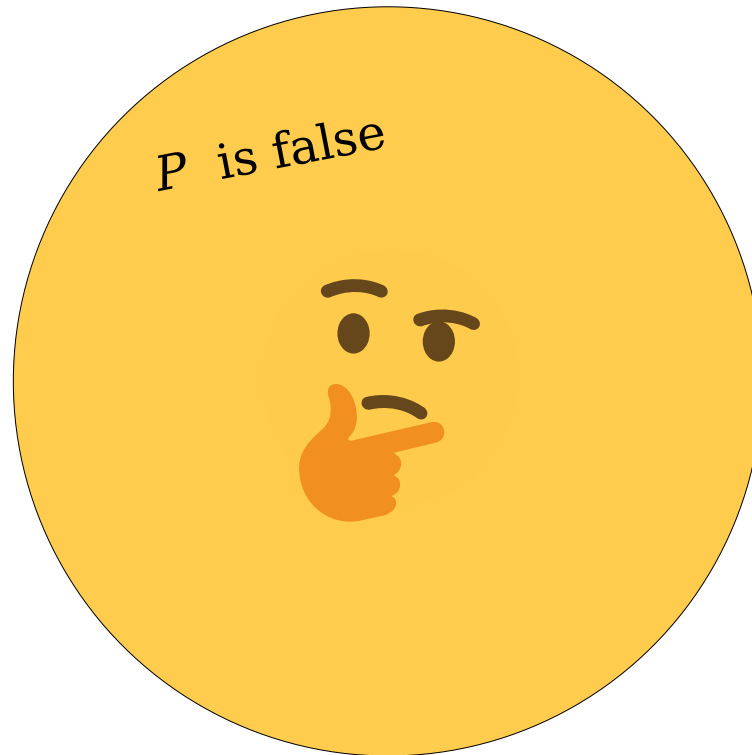
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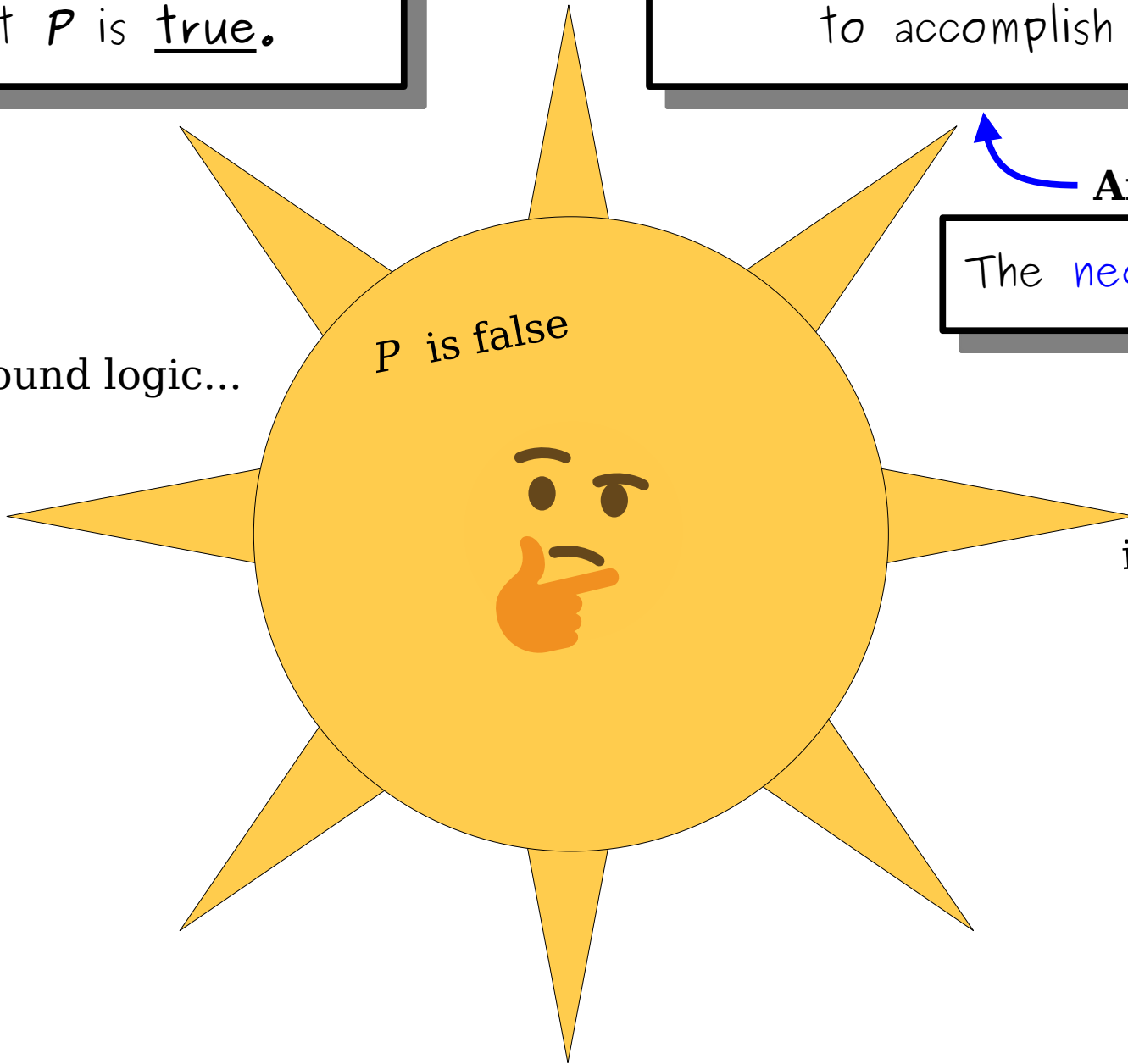
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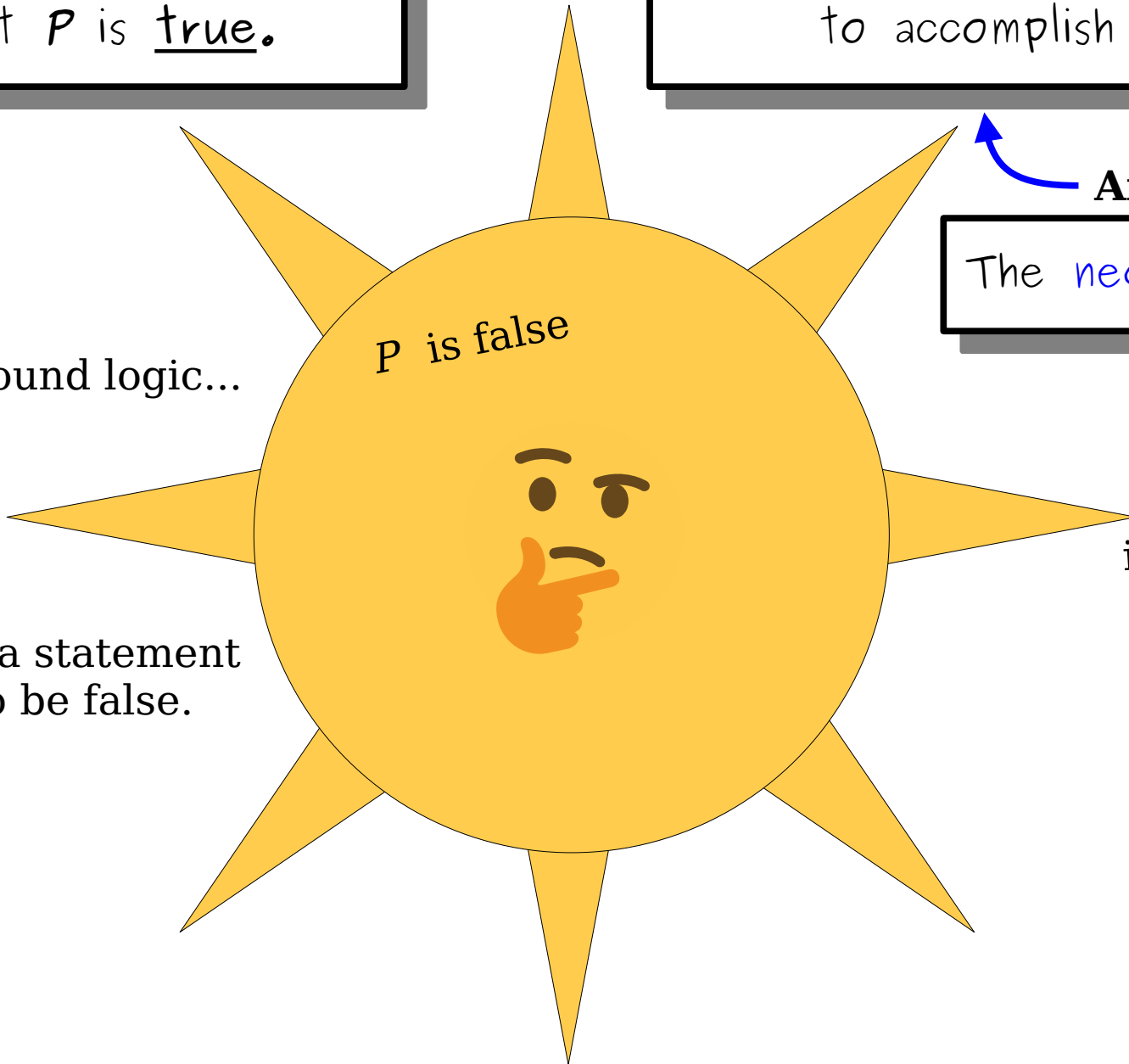
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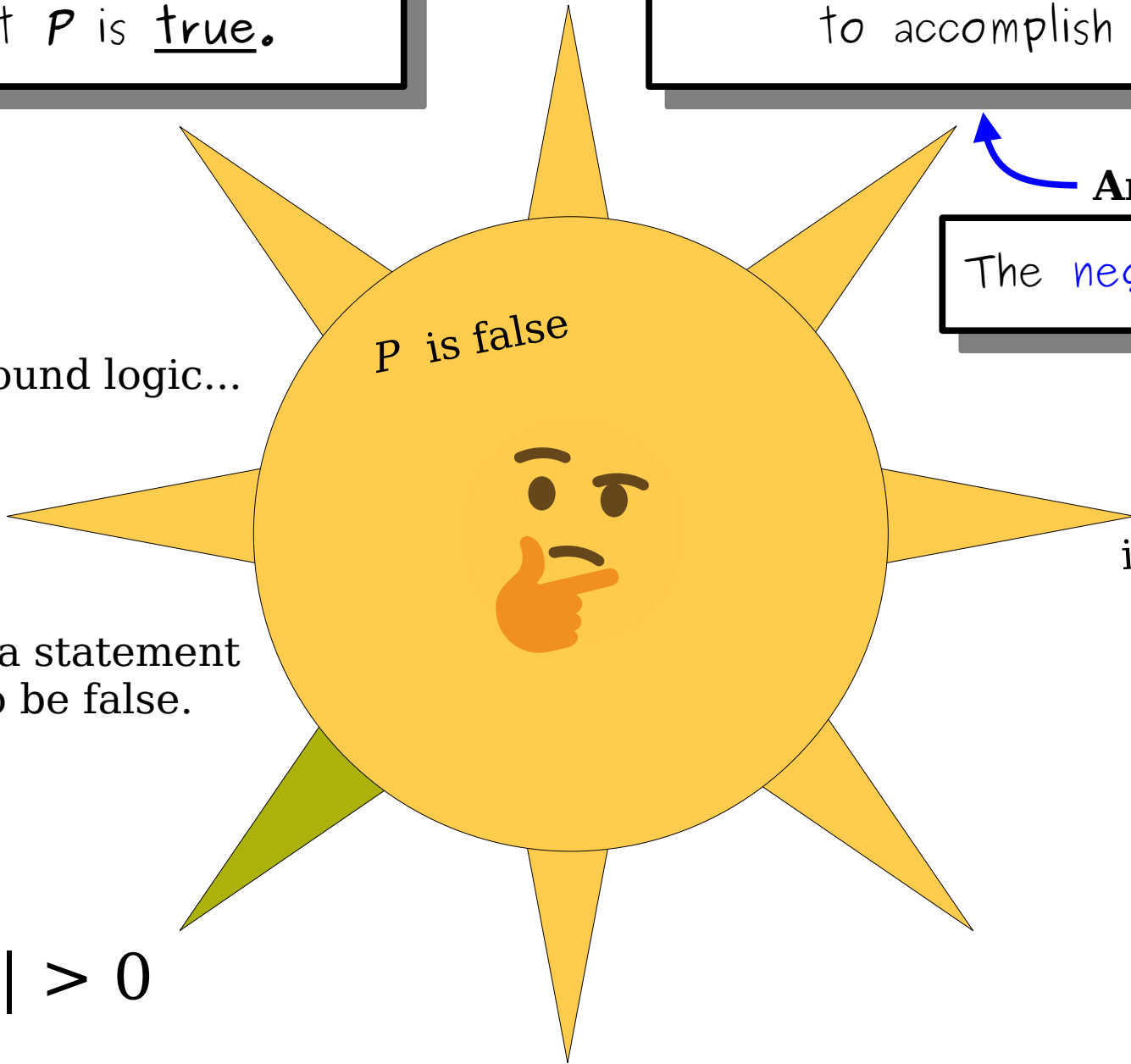
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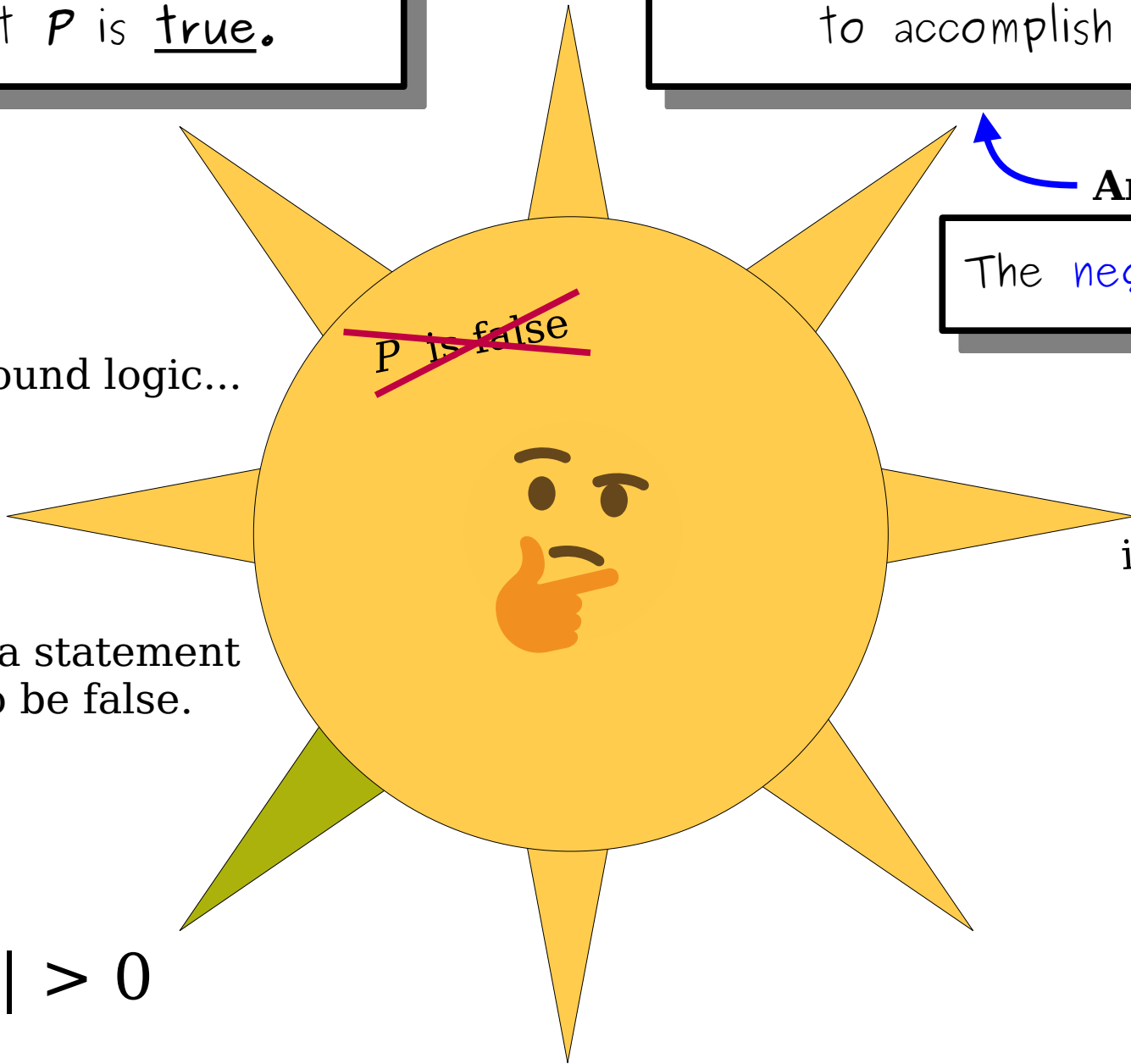
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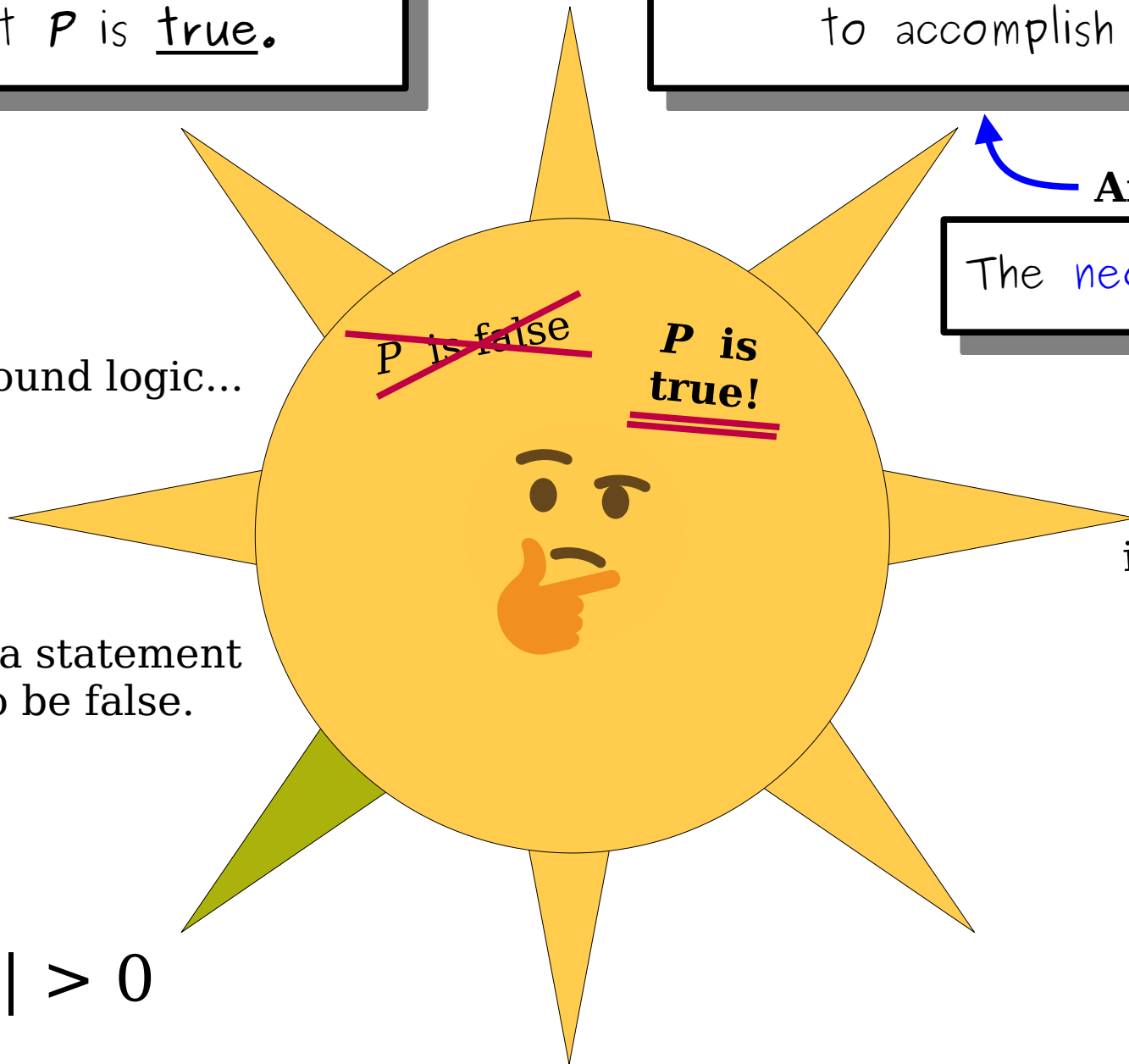
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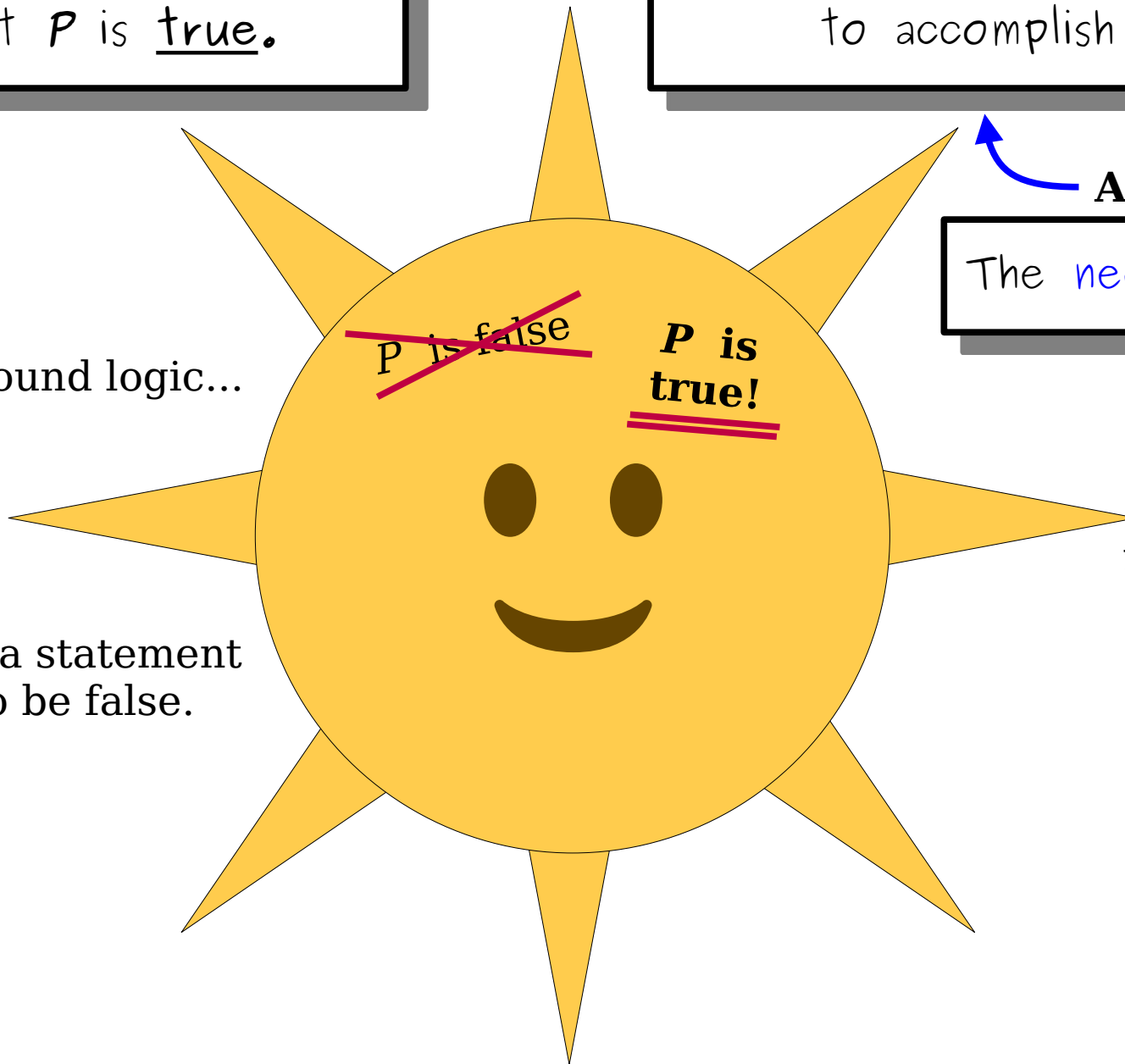
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Proof by Contradiction

- **Key Idea:** Prove a statement P is true by showing that it isn't false.
- First, assume that P is false. The goal is to show that this assumption is silly.
- Next, show this leads to an impossible result.
 - For example, we might have that $1 = 0$, that $x \in S$ and $x \notin S$, that a number is both even and odd, etc.
- Finally, conclude that since P can't be false, we know that P must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We've seen sets of many different cardinalities:
 - $|\emptyset| = 0$
 - $|\{1, 2, 3\}| = 3$
 - $|\{ n \in \mathbb{N} \mid n < 137 \}| = 137$
 - $|\mathbb{N}| = \aleph_0$.
 - $|\wp(\mathbb{N})| > |\mathbb{N}|$
- These span from the finite up through the infinite.
- **Question:** Is there a “largest” set? That is, is there a set that's bigger than every other set?

Theorem: There is no largest set.

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Proof:

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To prove this statement by contradiction,
we're going to assume its negation.

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What is the negation of the statement
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One option: "there is a largest set."

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S .

To prove this statement by contradiction,
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What is the negation of the statement
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Now, consider the set $\wp(S)$.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S .

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S .

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Another Example

Latin Squares

- A **Latin square** is an $n \times n$ grid filled with the numbers 1, 2, ..., n such that every number appears in each row and each column exactly once.

1	2	3
2	3	1
3	1	2

1	3	4	2
4	2	1	3
2	1	3	4
3	4	2	1

1	3	5	2	4
3	1	4	5	2
4	5	2	3	1
5	2	1	4	3
2	4	3	1	5

3	2	1	4	5	6
2	4	6	1	3	5
5	6	4	3	2	1
4	1	5	2	6	3
6	3	2	5	1	4
1	5	3	6	4	2

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3	1	4	5	2
4	5	2	3	1
5	2	1	4	3
2	4	3	1	5

3	2	1	4	5	6
2	4	6	1	3	5
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4	1	5	2	6	3
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1	3	4	5	2
4	2	5	3	1
3	4	2	1	5

Latin Squares

- Notice anything about what's on the main diagonals of these symmetric Latin squares?
- **Theorem:** Every odd-sized symmetric Latin square has every number $1, 2, \dots, n$ on its main diagonal.

1	2	3
2	3	1
3	1	2

1	2	3	4	5
2	5	4	1	3
3	4	2	5	1
4	1	5	3	2
5	3	1	2	4

3	2	5	1	4
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Theorem: Every symmetric Latin square of odd size $n \times n$ has each of the numbers $1, 2, \dots, n$ on its main diagonal.

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One option:

There is a symmetric Latin square of odd size $n \times n$ that does not have one of the numbers $1, 2, \dots, n$ on its main diagonal.

Theorem: Every symmetric Latin square of odd size $n \times n$ has each of the numbers $1, 2, \dots, n$ on its main diagonal.

Proof: Assume for the sake of contradiction that there is a symmetric Latin square of odd size $n \times n$ that does not have one of the numbers $1, 2, 3, \dots, n$ on its main diagonal.

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(Intermission)

Time-Out for Announcements!

Problem Set One

- Pset1 goes out today. It's due next Friday at 1 PM.
 - Explore the language of set theory and better intuit how it works.
 - Learn more about the structure of mathematical proofs.
 - Write your first “freehand” proofs based on your experiences.
- As always, ***start early***, and reach out if you have any questions!

Office Hours

- Office hours begin on Monday!
- Feel free to stop on by office hours – no appointment needed.
 - Check out the *Guide to Office Hours* for more information about how our OH system works.
 - The OH calendar is now available on the course website.
- It is ***completely normal*** in this class to need to get help from time to time.

PolLEV Begins Monday

- Come prepared to log into PolLEV. We'll give instructions on Monday, but you'll need a device connected to the wifi.
- We'll use PolLEV to record attendance/participation this quarter.
- Our expectation is that those participating in PolLEV are ***physically present*** in the classroom.



Working in Pairs

- Starting with Problem Set One, you are allowed to work either individually or in pairs.
 - Each pair should make a **single joint submission**.
- We have advice about how to work effectively in pairs up on the course website; check the “Guide to Partners.”
- Want to work in a pair, but don’t know who to work with? Fill out our [**Problem Set Matchmaking Google form**](#) (linked on Ed) by 11:59 PM tonight (Friday), and we’ll connect you with a partner sometime on Saturday.

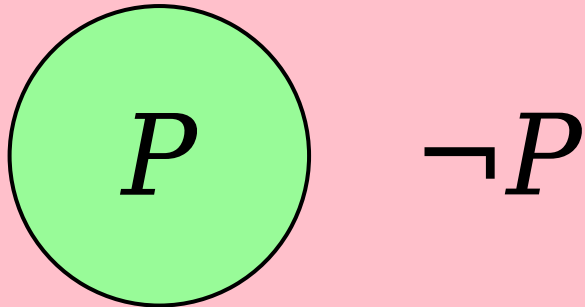
Readings for Today

- First is the ***Proofwriting Checklist***. It contains information about style expectations for proofs. We'll be using this when grading, so be sure to read it over.
- Next is the ***Guide to Office Hours***, which talks about how our office hours work and how to make the most effective use of them.
- Finally is the ***Guide to LaTeX***, which explains how to use LaTeX to typeset your problem sets in a way that's so beautiful it will bring tears to your eyes.
- Look for the seedling emoji in website menus for an indication of what's new each week.

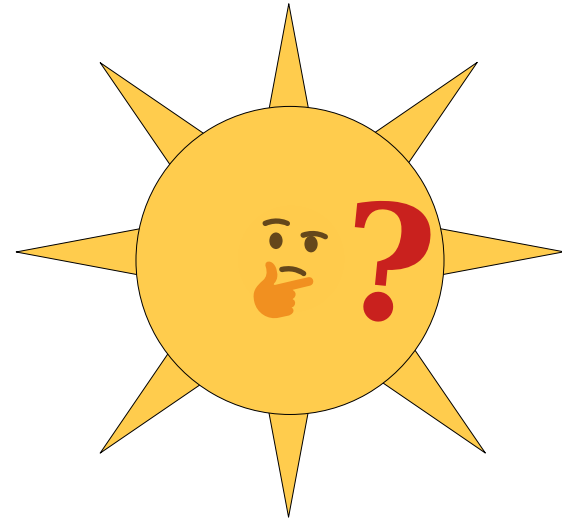
(the lights flash in the atrium)

Back to CS103!

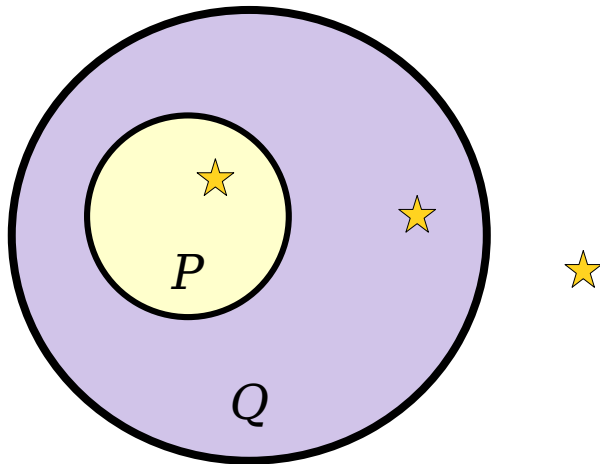
Indirect Proofs: A Story in Four Acts



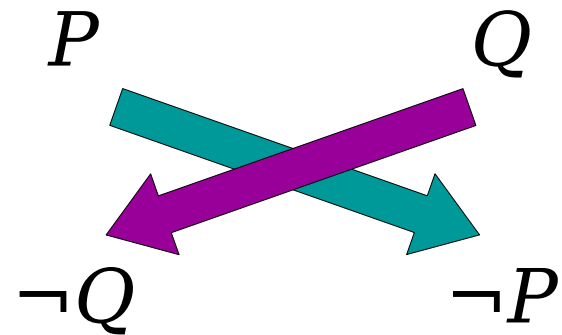
Logical Negation



Proof by Contradiction

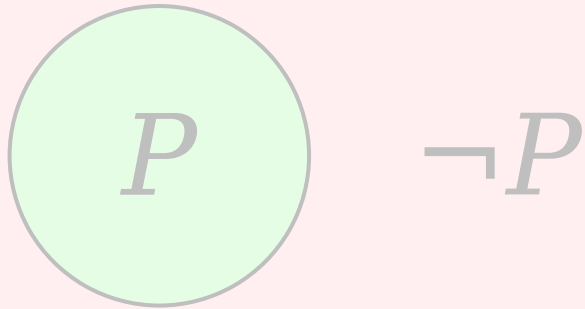


Logical Implication

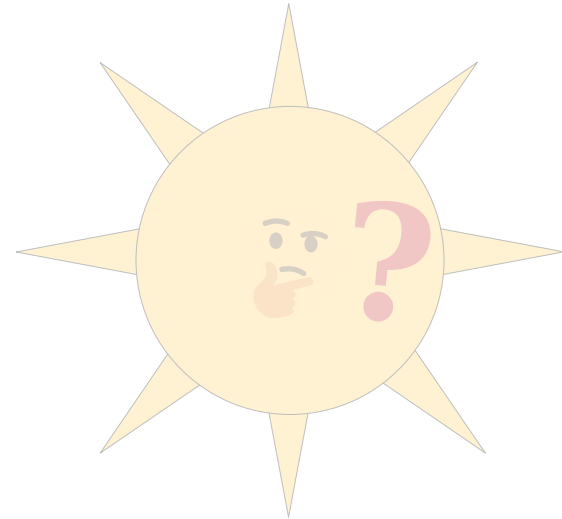


Proof by Contrapositive

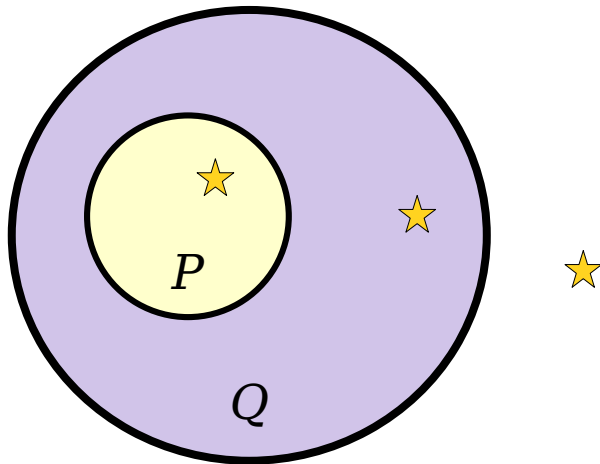
Indirect Proofs: A Story in Four Acts



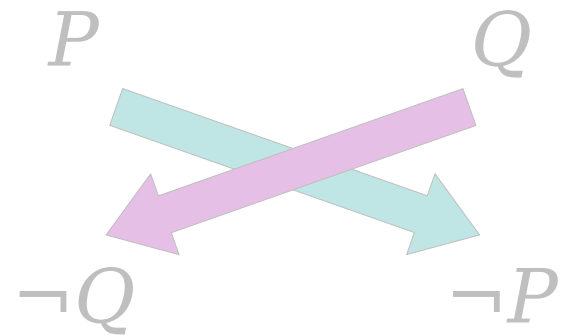
Logical Negation



Proof by Contradiction



Logical Implication



Proof by Contrapositive

Act III

Logical Implication

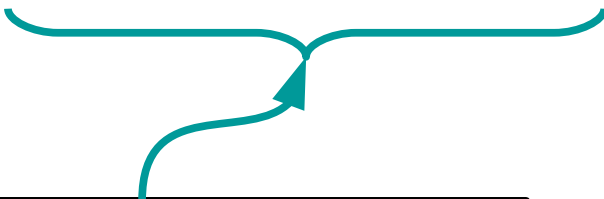
An *implication* is a statement of the form
“If P is true, then Q is true.”

An *implication* is a statement of the form
“If P is true, then Q is true.”

If n is an even integer, then n^2 is an even integer.

An ***implication*** is a statement of the form
“If P is true, then Q is true.”

If n is an even integer, then n^2 is an even integer.



This part of the implication is called the *antecedent*.



This part of the implication is called the *consequent*.

An **implication** is a statement of the form
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If m and n are odd integers, then $m+n$ is even.

An ***implication*** is a statement of the form
“If P is true, then Q is true.”

If n is an even integer, then n^2 is an even integer.

If m and n are odd integers, then $m+n$ is even.

If you like the way you look that much,
then you should go and love yourself.

An **implication** is a statement of the form
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If a flying pig bursts into the room and sings a pitch-perfect version of the national anthem, then Sean will throw cookies to the class.

An *implication* is a statement of the form
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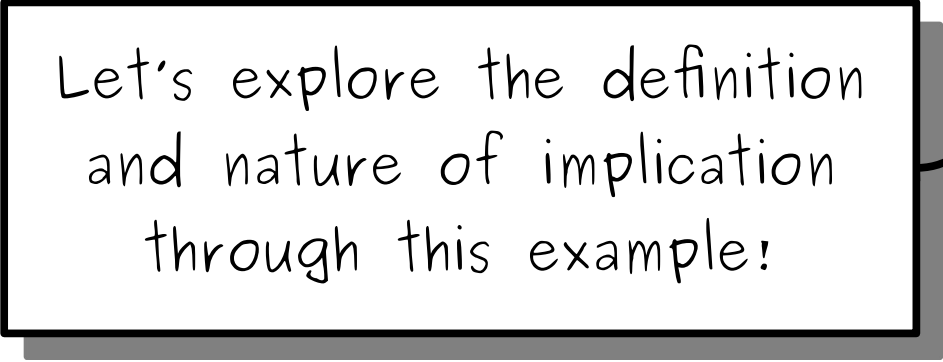
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Let's explore the definition and nature of implication through this example!

An **implication** is a statement of the form
“If P is true, then Q is true.”

If a flying pig bursts into the room and sings a pitch-perfect version of the national anthem,
then Sean will throw cookies to the class.

Let's explore the definition
and nature of implication
through this example!



“If P , then Q .”

An **implication** is a statement of the form
“If P is true, then Q is true.”

If a flying pig bursts into the room and sings a pitch-perfect version of the national anthem, then Sean will throw cookies to the class.

Let's explore the definition and nature of implication through this example!

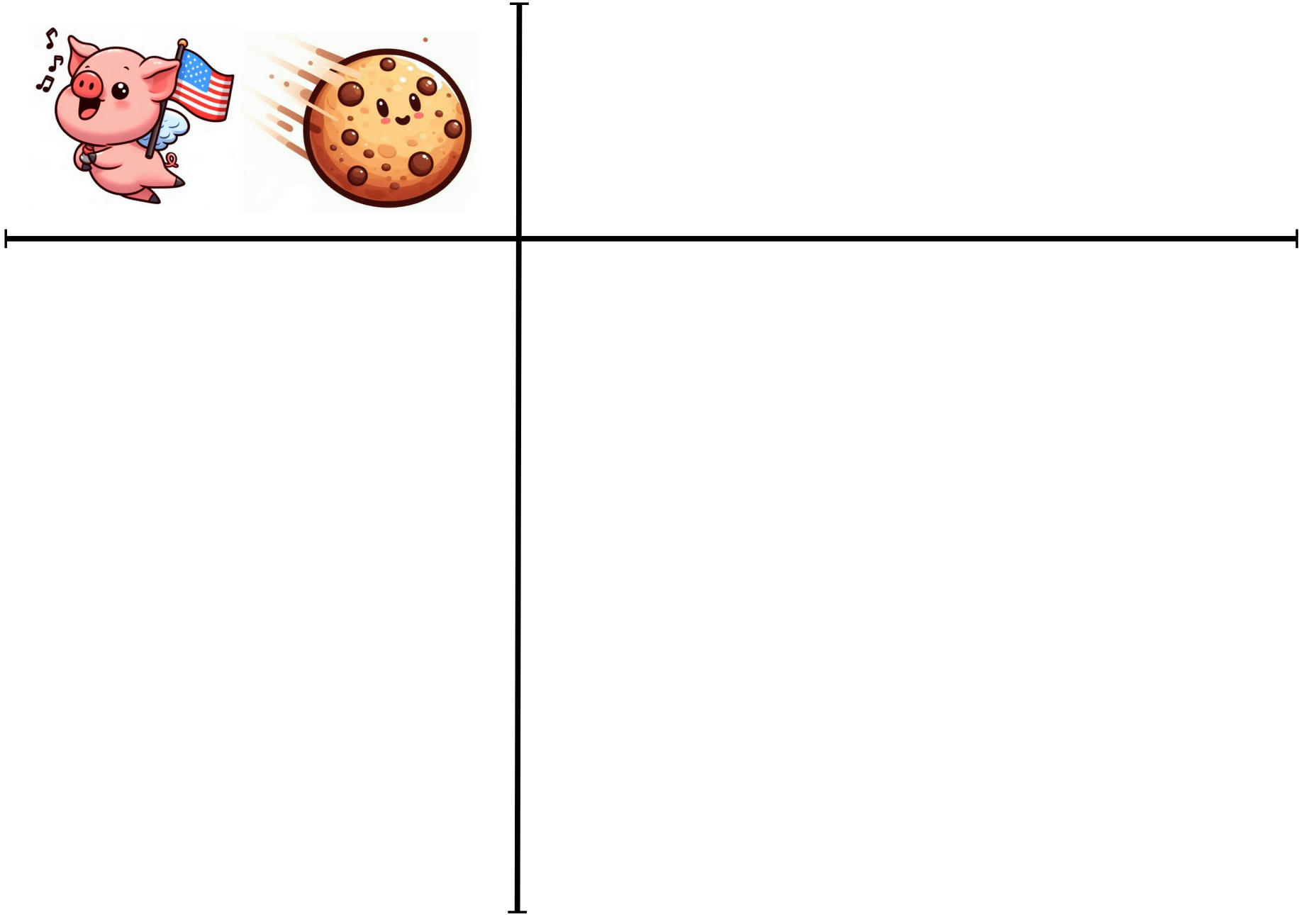
“If  , then  .”

An **implication** is a statement of the form
“If P is true, then Q is true.”











What is the status of our
“if , then ” contract?



What is the status of our
“if , then ” contract?





What is the status of our
“if , then ” contract?





What is the status of our
“if , then ” contract?



contract is not violated 



What is the status of our
“if , then ” contract?



contract is not violated 



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“if , then ” contract?



contract is not violated 





What is the status of our
“if , then  ” contract?



contract is not violated 





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contract **is** violated 



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contract is not violated 



contract **is** violated 



contract is not violated 



contract is not violated 



What is the status of our
“if , then ” contract?

This one often surprises people!
It's part of our definition of
implication and diverges from
how conditional statements work
in code.

✗

✓

contract is not violated ✓



What is the status of our
“if , then ” contract?

This one reveals how to
negate an implication!

contract is not violated ✓

contract **is** violated 😡

contract is not violated ✓

contract is not violated ✓

What is the status of our
“if , then ” contract?

This one reveals how to
negate an implication!

contract is not violated ✓



contract **is** violated 😡



contract is not violated ✓

The only time “if P , then Q ”
is false is when P is true and
 Q is false.



violated ✓

What Implications Mean

**“If there's a rainbow in the sky,
then it's raining somewhere.”**

- In mathematics, implication is directional.
 - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implications only say something about the consequent when the antecedent is true.
 - If there's no rainbow, it doesn't mean there's no rain.
- In mathematics, implication says nothing about causality.
 - Rainbows do not cause rain.

What Implications Mean

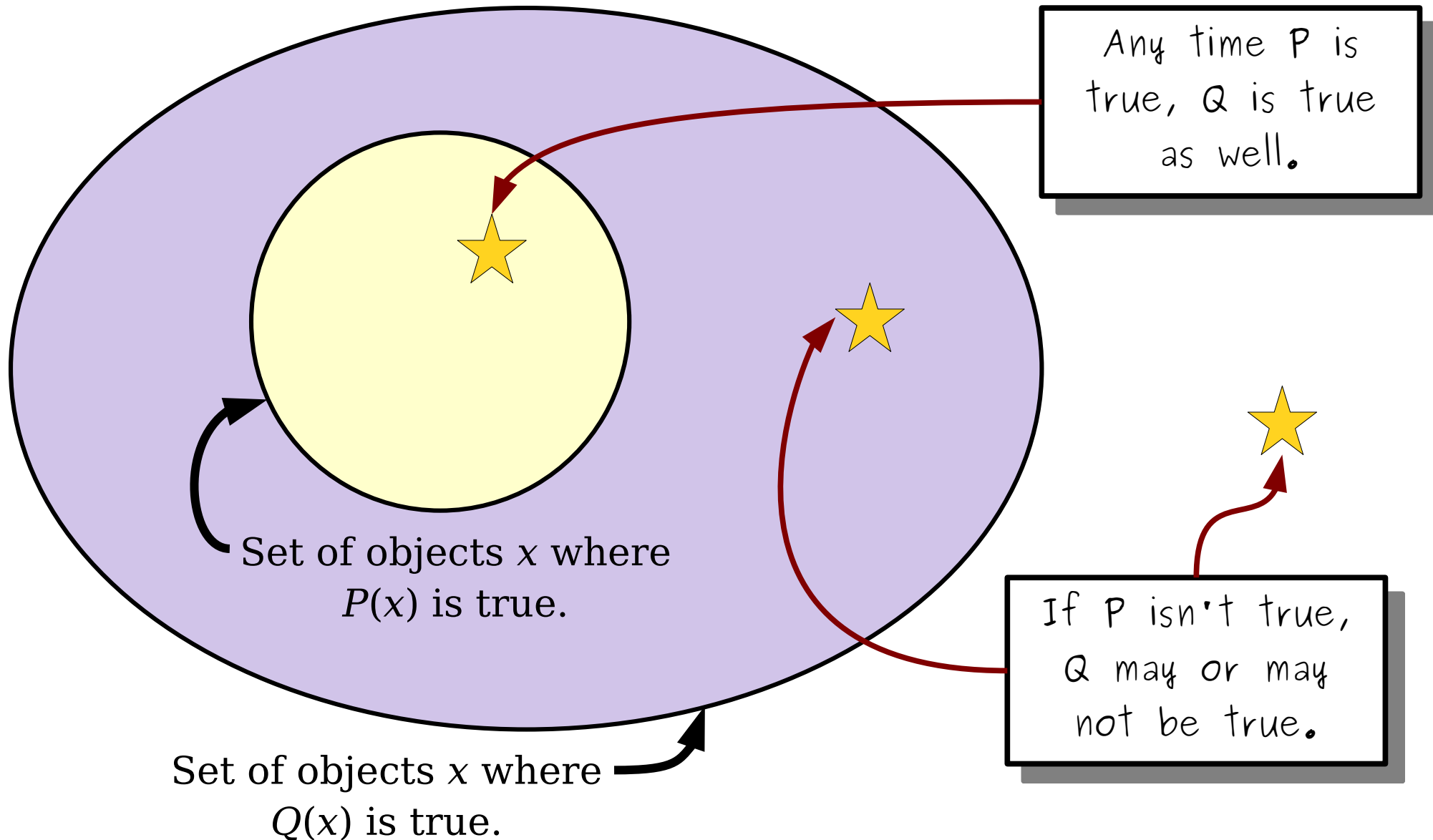
- In mathematics, a statement of the form

For any x , if $P(x)$ is true, then $Q(x)$ is true

means that any time you find an object x where $P(x)$ is true, you will see that $Q(x)$ is also true (for that same x).

- There is no discussion of causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is also true.

Implication, Diagrammatically



Negating an Implication

Consider once again the

“if , then ”
contract.

Question: What has to happen for this contract to be broken?

Answer: A flying pig sings the national anthem, but Sean doesn't throw cookies to the class.



What is the status of our
“if , then ” contract?



contract is not violated 



contract **is** violated 



contract is not violated 



contract is not violated 



What is the status of our
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contract is not violated 



contract **is** violated 



contract is not violated 



contract is not violated 

Negating an Implication

The negation of the statement

“If P , then Q .”

is the statement

“ P is true, and Q is false.”

***The negation of an implication
is not an implication!***

Negating an Implication

The negation of the statement

**“For any x , if $P(x)$ is true,
then $Q(x)$ is true”**

is the statement

**“There is at least one x where
 $P(x)$ is true and $Q(x)$ is false.”**

***The negation of an implication
is not an implication!***

Negating an Implication

The negation of the statement

“For any x , if $P(x)$ is true,
then $Q(x)$ is true”

is the statement

“There is at least one x where
 $P(x)$ is true and $Q(x)$ is false.”

***The negation of an implication
is not an implication!***

Key take-away!

Negating an Implication

- Example:
 - **If x is a puppy, then I love x .** 
 - **If x is a puppy, then I don't love x .** 
- These statements are **not** negations of one another!
- Consider this scenario:
 - There is some puppy that I love, and there is some puppy that I don't love.
 - Both implications are **false** in this case.
- The negation of the first one is:
 - **There is a puppy that I do not love.**

Negating a universal statement:

For all x , $P(x)$ is true.



There is an x where
 $P(x)$ is false.

Negating an existential statement:

There exists an x where
 $P(x)$ is true.



For all x , $P(x)$ is false.

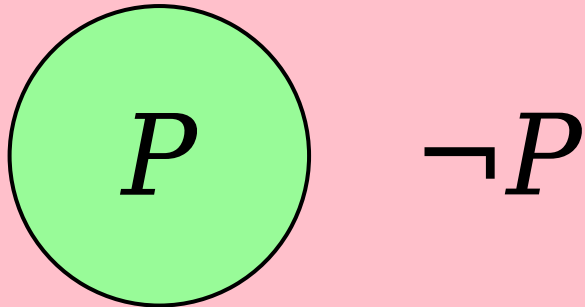
Negating an implication:

For every x , if $P(x)$ is true,
then $Q(x)$ is true.

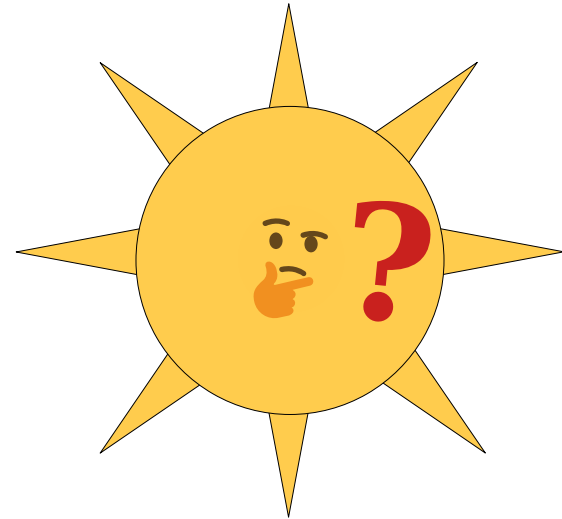


There is an x where $P(x)$ is
true and $Q(x)$ is false.

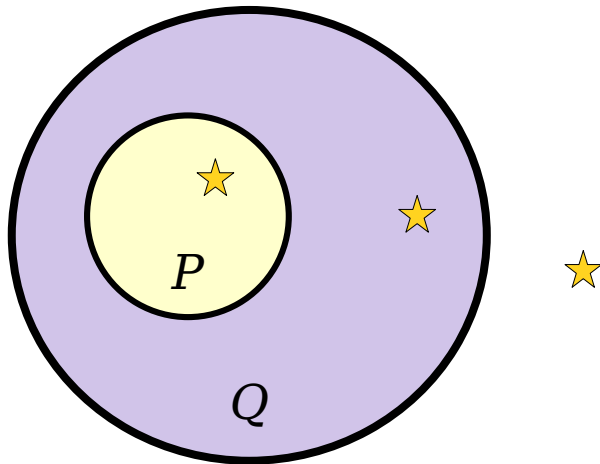
Indirect Proofs: A Story in Four Acts



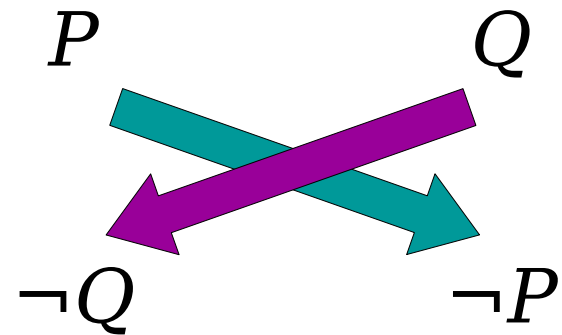
Logical Negation



Proof by Contradiction

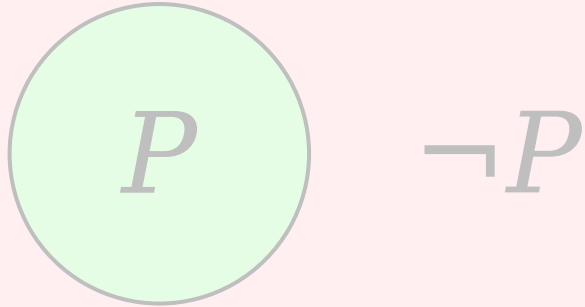


Logical Implication

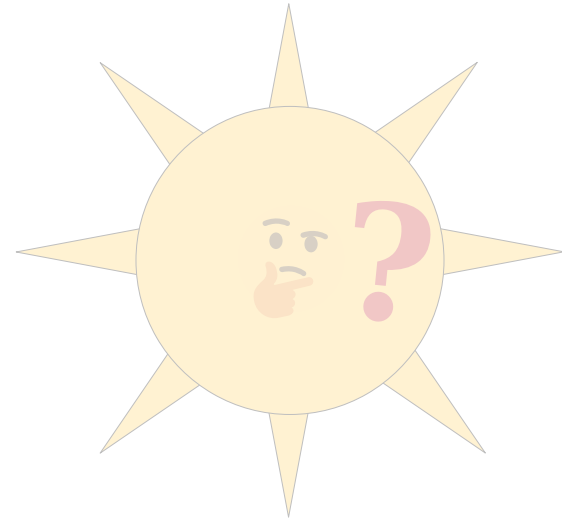


Proof by Contrapositive

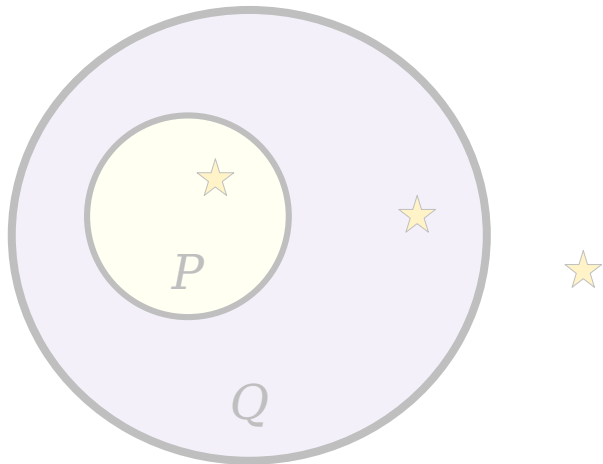
Indirect Proofs: A Story in Four Acts



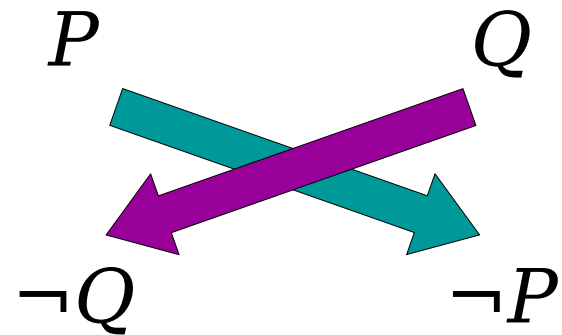
Logical Negation



Proof by Contradiction



Logical Implication



Proof by Contrapositive

Act IV

Proof by Contrapositive

<i>P</i>	<i>Q</i>	If <i>P</i>, then <i>Q</i>.	<i>Q</i> is false.	<i>P</i> is false.	If <i>Q</i> is false, then <i>P</i> is false.
✓	✓	✓	✗	✗	✓
✓	✗	✗	✓	✗	✗
✗	✗	✓	✓	✓	✓
✗	✓	✓	✗	✓	✓

P

Q

If P ,
then Q .

Q is
false.

P is
false.

If Q is
false, then
 P is false.

“If Q is false, then P is false”

is the *contrapositive* of

“If P is true, then Q is true.”

To prove the statement

“if P is true, then Q is true,”

you can choose to instead prove the equivalent statement

“if Q is false, then P is false,”

if that seems easier.

This is called a ***proof by contrapositive***.

The Contrapositive

- The **contrapositive** of the implication

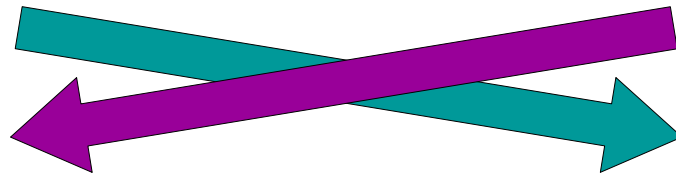
If **P is true**, then **Q is true**

is the implication

If **Q is false**, then **P is false**.

- The contrapositive of an implication means exactly the same thing as the implication itself.

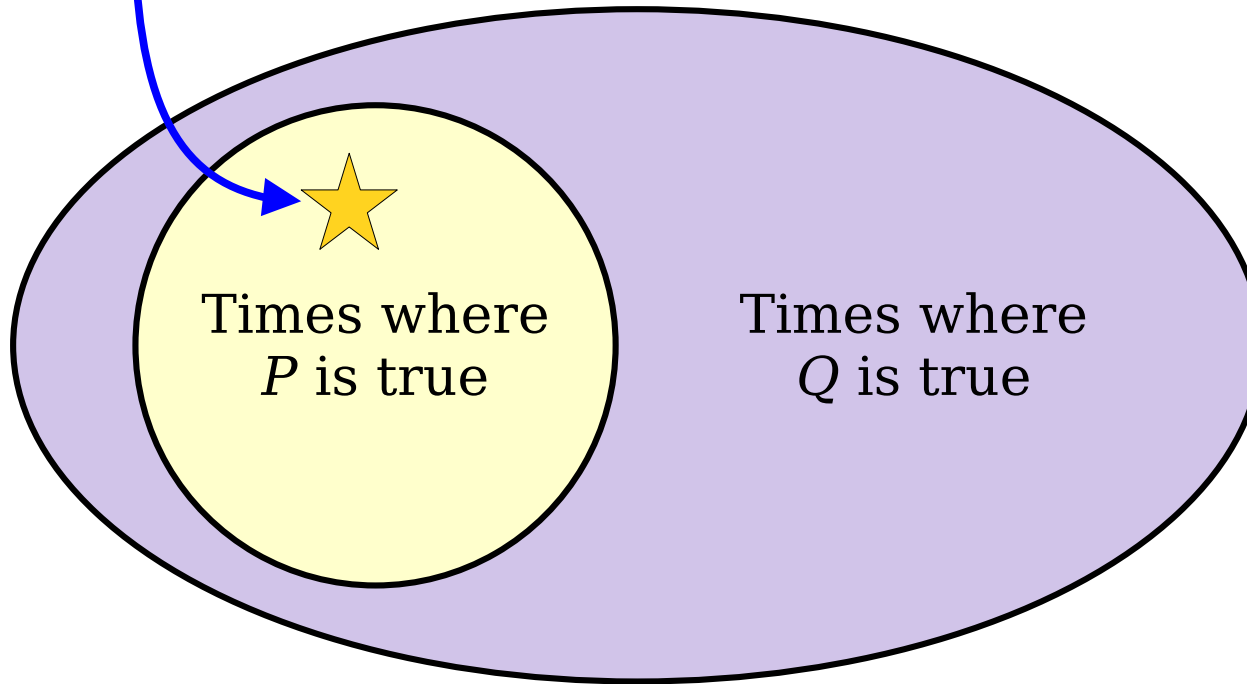
If it's a puppy, then I love it.



If I don't love it, then it's not a puppy.

Anything inside this inner bubble is also inside the outer bubble.

Anything outside this outer bubble is outside the inner bubble.



If P is true, then Q is true.

If Q is false, then P is false.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Proof: We will prove the contrapositive of this statement

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Proof: We will prove the contrapositive of this statement

This is a courtesy to the reader and says "heads up! we're not going to do a regular old-fashioned direct proof here."

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement,

What is the contrapositive of this statement?

if n^2 is even, then n is even.

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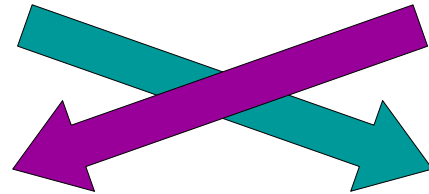
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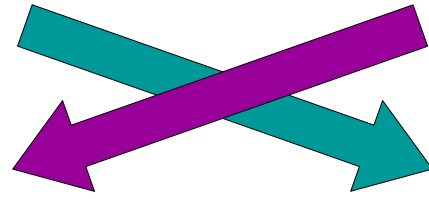


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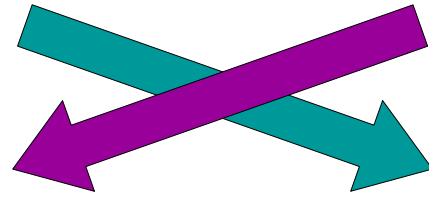
If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

What is the contrapositive of this statement?

if n^2 is even, then n is even.



If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that **if n is odd, then n^2 is odd**.

We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

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Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$.

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We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$n^2 = (2k + 1)^2$$

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$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

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Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

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From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd.

We know
integer
us that

The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.
2. Explicitly state the contrapositive of what we want to prove.
3. Go prove the contrapositive.

From th
(namely
means t
to show. ■

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

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From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. That means that n^2 is odd, which is what we needed to show. ■

Biconditionals

- The previous theorem, combined with what we saw on Wednesday, tells us the following:

For any integer n , if n is even, then n^2 is even.

For any integer n , if n^2 is even, then n is even.

- These are two different implications, each going the other way.
- We use the phrase ***if and only if*** to indicate that two statements imply one another.
- For example, we might combine the two above statements to say
for any integer n : n is even if and only if n^2 is even.

Proving Biconditionals

- To prove a theorem of the form

P if and only if Q ,

you need to prove two separate statements.

- First, that if P is true, then Q is true.
- Second, that if Q is true, then P is true.
- You can use any proof techniques you'd like to show each of these statements.
 - In our case, we used a direct proof for one and a proof by contrapositive for the other.

What We Learned

- ***How do you negate formulas?***
 - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.
- ***What's a proof by contradiction?***
 - It's a proof of a statement P that works by showing that P cannot be false.
- ***What's an implication?***
 - It's statement of the form “if P , then Q ,” and states that if P is true, then Q is true.
- ***What is a proof by contrapositive?***
 - It's a proof of an implication that instead proves its contrapositive.
 - (The contrapositive of “if P , then Q ” is “if not Q , then not P .”)

Your Action Items

- ***Read “Guide to Office Hours,” the “Proofwriting Checklist,” and the “Guide to LaTeX.”***
 - There’s a lot of useful information there. In particular, be sure to read the Proofwriting Checklist, as we’ll be working through this checklist when grading your proofs!
- ***Start working on PS1.***
 - At a bare minimum, read over it to see what’s being asked. That’ll give you time to turn things over in your mind this weekend.

Next Time

- ***Mathematical Logic***
 - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
 - Reasoning about simple statements.
- ***Propositional Equivalences***
 - Simplifying complex statements.

Appendix:
Proving Implications by Contradiction

Proving Implications

- Suppose we want to prove this implication:

If ***P* is true**, then ***Q* is true**.

- We have three options available to us:
 - ***Direct Proof:***
 - ***Proof by Contrapositive.***
 - ***Proof by Contradiction.***

Proving Implications

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... what does this look like?

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What is the negation of our theorem?

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what the negation of the original statement is.
3. Say you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

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- ***Proof by Contrapositive.***

Assume ***Q is false***, then prove that ***P is false***.

- ***Proof by Contradiction.***

Assume ***P is true*** and ***Q is false***,
then derive a contradiction.